

Icosahedron, exceptional singularities and modular forms

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Abstract

We find that the equation of E_8 -singularity possesses two distinct symmetry groups and modular parametrizations. One is the classical icosahedral equation with icosahedral symmetry, the associated modular forms are theta constants of order five. The other is given by the group $\mathrm{PSL}(2, 13)$, the associated modular forms are theta constants of order 13. As a consequence, we show that E_8 is not uniquely determined by the icosahedron. This solves a problem of Brieskorn in his ICM 1970 talk on the mysterious relation between exotic spheres, the icosahedron and E_8 . Simultaneously, it gives a counterexample to Arnold's A, D, E problem, and this also solves the other related problem on the relation between simple Lie algebras and Platonic solids. Moreover, we give modular parametrizations for the exceptional singularities Q_{18} , E_{20} and $x^7 + x^2y^3 + z^2 = 0$ by theta constants of order 13, the second singularity provides a new analytic construction of solutions for the Fermat-Catalan conjecture and gives an answer to a problem dating back to the works of Klein.

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1. Introduction

The icosahedron, and the other regular solids, have been known as the Platonic solids since the time of the ancient Greeks and have played an important role in the development of mathematics. Moreover they turn out to have unexpected relations to many other topics in mathematics. Some of the beauty and fascination associated with

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the icosahedron are given by the following correspondence between the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the *ADE* type Dynkin diagrams (see [Sl2]):

Group	Root system	
the cyclic group of order n	A_{n-1}	
the binary dihedral group of order $4n$	D_{n+2}	
the binary tetrahedral group of order 24	E_6	(1.1)
the binary octahedral group of order 48	E_7	
the binary icosahedral group of order 120	E_8	

This correspondence appears in many branches of mathematics and physics, such as singularity theory (see [Ar1], [Ar3], [Br3], [Sl1], [Sl2], [Dur]), modular forms and arithmetic (see [Hi3], [Na], [Se], [Du]), algebraic geometry (see [B], [DV], [GV], [IN], [R]), algebra (see [DM]), representation theory (see [Ko1], [Ko2], [Mc1], [Mc2], [N]), Lie groups (see [V]), geometry (see [At], [H]), geometric topology (see [KS]), differential geometry (see [Kr]), conformal field theory and subfactors (see [CIZ], [J1], [J2], [J3], [Z]) and string theory (see [W]). A connection between the left column and the right column in (1.1) is given by the singularities (see [Mc1]). In fact, the relation between simple singularities and simple Lie groups is one of the most beautiful discoveries in mathematics and it will lead back to the classification of Platonic solids (see [B]). Starting with the polynomial invariants of the finite subgroup of $\mathrm{SL}(2, \mathbb{C})$, a surface is defined from the single syzygy which relates the three polynomials in two variables. This surface has a singularity at the origin; the singularity can be resolved by constructing a smooth surface which is isomorphic to the original one except for a set of component curves which form the pre-image of the origin. The components form a Dynkin curve and the matrix of their intersections is the negative of the Cartan matrix for the appropriate Lie algebra. The Dynkin curve is the dual of the Dynkin graph. For example, if Γ is the binary icosahedral group, the corresponding Dynkin curve is that of E_8 , and $\mathbb{C}^2/\Gamma \subset \mathbb{C}^3$ is the set of zeros of the equation $x^2 + y^3 + z^5 = 0$ which can be parametrized by theta constants of order five (see section two for more details). The link of this E_8 -singularity, the Poincaré homology 3-sphere (see [KS]), has a higher dimensional lifting: $z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0$, $\sum_{i=1}^5 z_i \bar{z}_i = 1$, $z_i \in \mathbb{C}$ ($1 \leq i \leq 5$), which is the Brieskorn description of one of Milnor's exotic 7-dimensional spheres. In fact, it is an exotic 7-sphere representing Milnor's standard generator of Θ_7 (see [Br1], [Br2], [Br3], [Br5], [Hi1], [Hi2], [Hi4], [HM], [Mi1], [Mi2], [Mi3] and [Mi4]).

We will study the following problems which are closely related to the E_8 -singularity or more generally, the exceptional singularities.

Problem 1. The mysterious relation between exotic spheres, the icosahedron and E_8 .

In his ICM 1970 talk [Br3], Brieskorn showed how to construct the singularity of type *ADE* directly from the simple complex Lie group of the same type. At the end of that paper [Br3] Brieskorn says: "Thus we see that there is a relation between exotic spheres, the icosahedron and E_8 . But I still do not understand why the regular polyhedra come

in.” In fact, even today there is some mystery in these connections of such different parts of mathematics (see [Gr]).

Problem 2. Arnold’s A, D, E problem, especially for the appearance of the icosahedron.

In [Ar2] and [Ar6], Arnold posed the following A, D, E problem: “The Dynkin diagrams A_k, D_k, E_6, E_7, E_8 appear unexpectedly at the solution of so different classification problems, as the classifications (1) of critical points of functions, (2) of regular polyhedra in \mathbb{R}^3 , (3) the category of linear spaces, (4) caustics, (5) wave fronts, (6) reflection groups, and (7) simple Lie groups. Some connections between these objects are known, but in the majority of cases (e.g. in the cases $(1) \Leftrightarrow (2) \Leftrightarrow (3)$) the coincidence of the answers to different problems has until now no explanation at all. The problem “ A, D, E ” consists in finding such a general classification theorem from which the solutions of all listed problems could be derived such that one could derive, let us say, the classification of simple stable singularities of caustics from the classification of regular polyhedra.” Moreover, in [Ar3] and [Ar5], Arnold pointed out that the link between the theory of singularities and the classification of regular polyhedra in three-dimensional Euclidean space is rather mysteriously, especially for the appearance of the icosahedron.

Problem 3. The relation between simple Lie algebras and Platonic solids.

At the end of [Mc1], McKay posed the following: “Would not the Greeks appreciate the result that the simple Lie algebras may be derived from the Platonic solids?”

Problem 4. The Fermat-Catalan conjecture, especially the search for an eleventh solution.

The analytic construction of solutions of certain natural Diophantine equations is a problem of central importance in number theory. In particular, it plays a central role in the study of the following conjecture which contains within it essentially Fermat’s Last Theorem and also the Catalan conjecture (see [DG] and [D]).

Fermat-Catalan Conjecture. *If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then the Diophantine equation $x^p + y^q = z^r$ in $x, y, z \in \mathbb{Z}$ with $\gcd(x, y, z) = 1$, $xyz \neq 0$ and p, q, r are positive integers has no solutions except the following:*

$$1 + 2^3 = 3^2, 2^5 + 7^2 = 3^4, 7^3 + 13^2 = 2^9, 2^7 + 17^3 = 71^2, 3^5 + 11^4 = 122^2,$$

$$17^7 + 76271^3 = 21063928^2, 1414^3 + 2213459^2 = 65^7, 9262^3 + 15312283^2 = 113^7,$$

$$43^8 + 96222^3 = 30042907^2, 33^8 + 1549034^2 = 15613^3.$$

There is a general approach to studying the equation $x^p + y^q = z^r$, which can be viewed as a kind of nonabelian descent (see [PSS]). The principal homogeneous spaces which arise in this descent are algebraic curves equipped with a covering map to the projective line which is unramified outside of $\{0, 1, \infty\}$ and is of signature (p, q, r) . This just means that the ramification indices of the covering at the points above 0, 1 and ∞

divide p , q and r respectively. By a beautiful application of ideas from classical invariant theory and modular forms, the analytic construction of solutions were given in [Be] and [E] for $(p, q, r) = (2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, and in [PSS] for $(p, q, r) = (2, 3, 7)$ which is the exceptional singularity E_{12} (see [Br4] and [BPR]). The integer solutions were then obtained from analytic construction of solutions (see [Be], [E] and [PSS]). A good starting point for the Fermat-Catalan Conjecture is to search for the eleventh solution.

In the present paper, we establish the invariant theory for $\text{PSL}(2, 13)$. Let us begin with the six-dimensional representation of the finite simple group $\text{PSL}(2, 13)$ of order 1092, which acts on the five-dimensional projective space $\mathbb{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbb{C} \ (i = 1, 2, 3, 4, 5, 6)\}$. This representation is defined over the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{13}})$. Put

$$S = -\frac{1}{\sqrt{13}} \begin{pmatrix} \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 \\ \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\ \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} \\ \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 \\ \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} \\ \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} \end{pmatrix}$$

and $T = \text{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5)$ where $\zeta = \exp(2\pi i/13)$. We have $S^2 = T^{13} = (ST)^3 = 1$. Let $G = \langle S, T \rangle$, then $G \cong \text{PSL}(2, 13)$. We construct some G -invariant polynomials in six variables z_1, \dots, z_6 which come from the Jacobian equation of degree fourteen and the exotic equation of degree fourteen. Let

$$w_\infty = 13\mathbb{A}_0^2, \quad w_\nu = (\mathbb{A}_0 + \zeta^\nu \mathbb{A}_1 + \zeta^{4\nu} \mathbb{A}_2 + \zeta^{9\nu} \mathbb{A}_3 + \zeta^{3\nu} \mathbb{A}_4 + \zeta^{12\nu} \mathbb{A}_5 + \zeta^{10\nu} \mathbb{A}_6)^2 \quad (1.2)$$

for $\nu = 0, 1, \dots, 12$, where the senary quadratic forms (quadratic forms in six variables) \mathbb{A}_j ($j = 0, 1, \dots, 6$) are given by

$$\begin{cases} \mathbb{A}_0 = z_1 z_4 + z_2 z_5 + z_3 z_6, \\ \mathbb{A}_1 = z_1^2 - 2z_3 z_4, \\ \mathbb{A}_2 = -z_5^2 - 2z_2 z_4, \\ \mathbb{A}_3 = z_2^2 - 2z_1 z_5, \\ \mathbb{A}_4 = z_3^2 - 2z_2 z_6, \\ \mathbb{A}_5 = -z_4^2 - 2z_1 z_6, \\ \mathbb{A}_6 = -z_6^2 - 2z_3 z_5. \end{cases} \quad (1.3)$$

Then w_∞, w_ν for $\nu = 0, \dots, 12$ form an algebraic equation of degree fourteen, which is just the Jacobian equation of degree fourteen, whose roots are these w_ν and w_∞ . On the other hand, set

$$\delta_\infty = 13^2 \mathbb{G}_0, \quad \delta_\nu = -13 \mathbb{G}_0 + \zeta^\nu \mathbb{G}_1 + \zeta^{2\nu} \mathbb{G}_2 + \dots + \zeta^{12\nu} \mathbb{G}_{12} \quad (1.4)$$

for $\nu = 0, 1, \dots, 12$, where the senary sextic forms (i.e., sextic forms in six variables) G_j ($j = 0, 1, \dots, 12$) are given by

$$\left\{ \begin{array}{l} G_0 = \mathbb{D}_0^2 + \mathbb{D}_\infty^2, \\ G_1 = -\mathbb{D}_7^2 + 2\mathbb{D}_0\mathbb{D}_1 + 10\mathbb{D}_\infty\mathbb{D}_1 + 2\mathbb{D}_2\mathbb{D}_{12} - 2\mathbb{D}_3\mathbb{D}_{11} - 4\mathbb{D}_4\mathbb{D}_{10} - 2\mathbb{D}_9\mathbb{D}_5, \\ G_2 = -2\mathbb{D}_1^2 - 4\mathbb{D}_0\mathbb{D}_2 + 6\mathbb{D}_\infty\mathbb{D}_2 - 2\mathbb{D}_4\mathbb{D}_{11} + 2\mathbb{D}_5\mathbb{D}_{10} - 2\mathbb{D}_6\mathbb{D}_9 - 2\mathbb{D}_7\mathbb{D}_8, \\ G_3 = -\mathbb{D}_8^2 + 2\mathbb{D}_0\mathbb{D}_3 + 10\mathbb{D}_\infty\mathbb{D}_3 + 2\mathbb{D}_6\mathbb{D}_{10} - 2\mathbb{D}_9\mathbb{D}_7 - 4\mathbb{D}_{12}\mathbb{D}_4 - 2\mathbb{D}_1\mathbb{D}_2, \\ G_4 = -\mathbb{D}_2^2 + 10\mathbb{D}_0\mathbb{D}_4 - 2\mathbb{D}_\infty\mathbb{D}_4 + 2\mathbb{D}_5\mathbb{D}_{12} - 2\mathbb{D}_9\mathbb{D}_8 - 4\mathbb{D}_1\mathbb{D}_3 - 2\mathbb{D}_{10}\mathbb{D}_7, \\ G_5 = -2\mathbb{D}_9^2 - 4\mathbb{D}_0\mathbb{D}_5 + 6\mathbb{D}_\infty\mathbb{D}_5 - 2\mathbb{D}_{10}\mathbb{D}_8 + 2\mathbb{D}_6\mathbb{D}_{12} - 2\mathbb{D}_2\mathbb{D}_3 - 2\mathbb{D}_{11}\mathbb{D}_7, \\ G_6 = -2\mathbb{D}_3^2 - 4\mathbb{D}_0\mathbb{D}_6 + 6\mathbb{D}_\infty\mathbb{D}_6 - 2\mathbb{D}_{12}\mathbb{D}_7 + 2\mathbb{D}_2\mathbb{D}_4 - 2\mathbb{D}_5\mathbb{D}_1 - 2\mathbb{D}_8\mathbb{D}_{11}, \\ G_7 = -2\mathbb{D}_{10}^2 + 6\mathbb{D}_0\mathbb{D}_7 + 4\mathbb{D}_\infty\mathbb{D}_7 - 2\mathbb{D}_1\mathbb{D}_6 - 2\mathbb{D}_2\mathbb{D}_5 - 2\mathbb{D}_8\mathbb{D}_{12} - 2\mathbb{D}_9\mathbb{D}_{11}, \\ G_8 = -2\mathbb{D}_4^2 + 6\mathbb{D}_0\mathbb{D}_8 + 4\mathbb{D}_\infty\mathbb{D}_8 - 2\mathbb{D}_3\mathbb{D}_5 - 2\mathbb{D}_6\mathbb{D}_2 - 2\mathbb{D}_{11}\mathbb{D}_{10} - 2\mathbb{D}_1\mathbb{D}_7, \\ G_9 = -\mathbb{D}_{11}^2 + 2\mathbb{D}_0\mathbb{D}_9 + 10\mathbb{D}_\infty\mathbb{D}_9 + 2\mathbb{D}_5\mathbb{D}_4 - 2\mathbb{D}_1\mathbb{D}_8 - 4\mathbb{D}_{10}\mathbb{D}_{12} - 2\mathbb{D}_3\mathbb{D}_6, \\ G_{10} = -\mathbb{D}_5^2 + 10\mathbb{D}_0\mathbb{D}_{10} - 2\mathbb{D}_\infty\mathbb{D}_{10} + 2\mathbb{D}_6\mathbb{D}_4 - 2\mathbb{D}_3\mathbb{D}_7 - 4\mathbb{D}_9\mathbb{D}_1 - 2\mathbb{D}_{12}\mathbb{D}_{11}, \\ G_{11} = -2\mathbb{D}_{12}^2 + 6\mathbb{D}_0\mathbb{D}_{11} + 4\mathbb{D}_\infty\mathbb{D}_{11} - 2\mathbb{D}_9\mathbb{D}_2 - 2\mathbb{D}_5\mathbb{D}_6 - 2\mathbb{D}_7\mathbb{D}_4 - 2\mathbb{D}_3\mathbb{D}_8, \\ G_{12} = -\mathbb{D}_6^2 + 10\mathbb{D}_0\mathbb{D}_{12} - 2\mathbb{D}_\infty\mathbb{D}_{12} + 2\mathbb{D}_2\mathbb{D}_{10} - 2\mathbb{D}_1\mathbb{D}_{11} - 4\mathbb{D}_3\mathbb{D}_9 - 2\mathbb{D}_4\mathbb{D}_8. \end{array} \right. \quad (1.5)$$

Here, the senary cubic forms (cubic forms in six variables) \mathbb{D}_j ($j = 0, 1, \dots, 12, \infty$) are given as follows:

$$\left\{ \begin{array}{l} \mathbb{D}_0 = z_1 z_2 z_3, \\ \mathbb{D}_1 = 2z_2 z_3^2 + z_2^2 z_6 - z_4^2 z_5 + z_1 z_5 z_6, \\ \mathbb{D}_2 = -z_6^3 + z_2^2 z_4 - 2z_2 z_5^2 + z_1 z_4 z_5 + 3z_3 z_5 z_6, \\ \mathbb{D}_3 = 2z_1 z_2^2 + z_1^2 z_5 - z_4 z_6^2 + z_3 z_4 z_5, \\ \mathbb{D}_4 = -z_2^2 z_3 + z_1 z_6^2 - 2z_4^2 z_6 - z_1 z_3 z_5, \\ \mathbb{D}_5 = -z_4^3 + z_3^2 z_5 - 2z_3 z_6^2 + z_2 z_5 z_6 + 3z_1 z_4 z_6, \\ \mathbb{D}_6 = -z_5^3 + z_1^2 z_6 - 2z_1 z_4^2 + z_3 z_4 z_6 + 3z_2 z_4 z_5, \\ \mathbb{D}_7 = -z_2^3 + z_3 z_4^2 - z_1 z_3 z_6 - 3z_1 z_2 z_5 + 2z_1^2 z_4, \\ \mathbb{D}_8 = -z_1^3 + z_2 z_6^2 - z_2 z_3 z_5 - 3z_1 z_3 z_4 + 2z_3^2 z_6, \\ \mathbb{D}_9 = 2z_1^2 z_3 + z_3^2 z_4 - z_5^2 z_6 + z_2 z_4 z_6, \\ \mathbb{D}_{10} = -z_1 z_3^2 + z_2 z_4^2 - 2z_4 z_5^2 - z_1 z_2 z_6, \\ \mathbb{D}_{11} = -z_3^3 + z_1 z_5^2 - z_1 z_2 z_4 - 3z_2 z_3 z_6 + 2z_2^2 z_5, \\ \mathbb{D}_{12} = -z_1^2 z_2 + z_3 z_5^2 - 2z_5 z_6^2 - z_2 z_3 z_4, \\ \mathbb{D}_\infty = z_4 z_5 z_6. \end{array} \right. \quad (1.6)$$

Then $\delta_\infty, \delta_\nu$ for $\nu = 0, \dots, 12$ form an algebraic equation of degree fourteen which is not the Jacobian equation of degree fourteen. We call it the exotic equation of degree fourteen. Let

$$\Phi_{12} = -\frac{1}{13 \cdot 52} \left(\sum_{\nu=0}^{12} \delta_\nu^2 + \delta_\infty^2 \right), \quad \Phi_{18} = \frac{1}{13 \cdot 6} \left(\sum_{\nu=0}^{12} \delta_\nu^3 + \delta_\infty^3 \right), \quad (1.7)$$

$$\Phi_{20} = \frac{1}{13 \cdot 25} \left(\sum_{\nu=0}^{12} w_\nu^5 + w_\infty^5 \right), \quad \Phi_{30} = -\frac{1}{13 \cdot 1315} \left(\sum_{\nu=0}^{12} \delta_\nu^5 + \delta_\infty^5 \right), \quad (1.8)$$

and $x_i(z) = \eta(z)a_i(z)$ ($1 \leq i \leq 6$), where

$$\begin{cases} a_1(z) := e^{-\frac{11\pi i}{26}\theta} \begin{bmatrix} \frac{11}{13} \\ 1 \end{bmatrix} (0, 13z), \\ a_2(z) := e^{-\frac{7\pi i}{26}\theta} \begin{bmatrix} \frac{7}{13} \\ 1 \end{bmatrix} (0, 13z), \\ a_3(z) := e^{-\frac{5\pi i}{26}\theta} \begin{bmatrix} \frac{5}{13} \\ 1 \end{bmatrix} (0, 13z), \\ a_4(z) := -e^{-\frac{3\pi i}{26}\theta} \begin{bmatrix} \frac{3}{13} \\ 1 \end{bmatrix} (0, 13z), \\ a_5(z) := e^{-\frac{9\pi i}{26}\theta} \begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix} (0, 13z), \\ a_6(z) := e^{-\frac{\pi i}{26}\theta} \begin{bmatrix} \frac{1}{13} \\ 1 \end{bmatrix} (0, 13z) \end{cases} \quad (1.9)$$

are theta constants of order 13 and $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi iz}$ is the Dedekind eta function which are all defined in the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Our main result is to show that there exists at least two kinds of symmetry groups associated with the equation of E_8 -singularity: one is the icosahedral group (the standard structure), the other is the group $\text{PSL}(2, 13)$ (the exotic structure).

Theorem 1.1 (Main Theorem 1). *The exotic structure on the E_8 -singularity is given by the following equations:*

$$\Phi_{20}^3 - \Phi_{30}^2 = 1728\Phi_{12}^5, \quad \Phi_{20}^3 - \Phi_{12}^2\Phi_{18}^2 = 1728\Phi_{12}^5, \quad (1.10)$$

where we use the abbreviation $\Phi_j = \Phi_j(x_1(z), \dots, x_6(z))$ for $j = 12, 18, 20$ and 30 . As polynomials in six variables z_1, \dots, z_6 , Φ_{12} , Φ_{18} , Φ_{20} and Φ_{30} are G -invariant polynomials.

As a consequence, Theorem 1.1 shows that E_8 is not uniquely determined by the icosahedron. This gives a negative answer to Problem 1, i.e., the icosahedron does not

necessarily appear in the triple (exotic spheres, icosahedron, E_8). The group $\text{PSL}(2, 13)$ can take its place and there is the other triple (exotic spheres, $\text{PSL}(2, 13)$, E_8). The higher dimensional liftings of these two distinct symmetry groups and modular interpretations on the equation of E_8 -singularity give the same Milnor's standard generator of Θ_7 . Simultaneously, Theorem 1.1 gives a counterexample to Problem 2. It shows that the equivalence between critical points of functions and regular polyhedra in \mathbb{R}^3 in Problem 2 is not true. In particular, the appearance of the icosahedron is not necessary. The group $\text{PSL}(2, 13)$ can take its place, which is the symmetry group of the regular maps $\{7, 3\}_{13}$, $\{13, 3\}_7$ or $\{13, 7\}_3$ (see [CoM], p.140). Theorem 1.1 also shows that the simple Lie algebra E_8 may not be necessarily derived from the icosahedron, which gives a negative answer to Problem 3. On the other hand, Theorem 1.1 provides an analytic construction of solutions for the Diophantine equation $x^p + y^q = z^r$ in the case $(p, q, r) = (2, 3, 5)$ which is different from the solutions given in [E].

Let

$$\Phi_{32} = -\frac{1}{13 \cdot 1840} \left(\sum_{\nu=0}^{12} w_{\nu}^8 + w_{\infty}^8 \right), \quad \Phi_{42} = \frac{1}{13 \cdot 226842} \left(\sum_{\nu=0}^{12} \delta_{\nu}^7 + \delta_{\infty}^7 \right), \quad (1.11)$$

$$\Phi_{44} = \frac{1}{13 \cdot 146905} \left(\sum_{\nu=0}^{12} w_{\nu}^{11} + w_{\infty}^{11} \right), \quad (1.12)$$

Theorem 1.2. (Main Theorem 2). *The exceptional singularities $Q_{18} : x^3 + y^8 + yz^2 = 0$, $E_{20} : x^3 + y^{11} + z^2 = 0$ and $x^7 + x^2y^3 + z^2 = 0$ can be endowed with the symmetry group G and have the following modular parametrizations:*

$$\left\{ \begin{array}{l} \Phi_{20}^3 \Phi_{12}^2 - \Phi_{42}^2 = 1728 \Phi_{12}^7, \\ \Phi_{32}^3 - \Phi_{12}^5 \Phi_{18}^2 = 1728 \Phi_{12}^8, \\ \Phi_{32}^3 - \Phi_{12}^3 \Phi_{30}^2 = 1728 \Phi_{12}^8, \\ \Phi_{32}^3 - \Phi_{12} \Phi_{42}^2 = 1728 \Phi_{12}^8, \\ \Phi_{44}^3 - \Phi_{12}^8 \Phi_{18}^2 = 1728 \Phi_{12}^{11}, \\ \Phi_{44}^3 - \Phi_{12}^6 \Phi_{30}^2 = 1728 \Phi_{12}^{11}, \\ \Phi_{44}^3 - \Phi_{12}^4 \Phi_{42}^2 = 1728 \Phi_{12}^{11}, \end{array} \right. \quad (1.13)$$

where we use the abbreviation $\Phi_j = \Phi_j(x_1(z), \dots, x_6(z))$ for $j = 32, 42$ and 44 . As polynomials in six variables z_1, \dots, z_6 , Φ_{12} , Φ_{18} , Φ_{20} , Φ_{30} , Φ_{32} , Φ_{42} and Φ_{44} are G -invariant polynomials.

Corollary 1.3. *The Diophantine equation $x^2 + y^3 = z^{11}$ has the following analytic construction of solutions:*

$$\left\{ \begin{array}{l} \Phi_{44}^3 - \Phi_{12}^8 \Phi_{18}^2 = 1728 \Phi_{12}^{11}, \\ \Phi_{44}^3 - \Phi_{12}^6 \Phi_{30}^2 = 1728 \Phi_{12}^{11}, \\ \Phi_{44}^3 - \Phi_{12}^4 \Phi_{42}^2 = 1728 \Phi_{12}^{11}, \end{array} \right. \quad (1.14)$$

where we use the abbreviation $\Phi_j = \Phi_j(x_1(z), \dots, x_6(z))$ for $j = 12, 18, 30, 42$ and 44 .

We can multiply the Diophantine equation $x^2 + y^3 = z^{11}$ with $\gcd(x, y, z) = 1$ by $3^{36}2^6$ to obtain $(-3^{12}2^2y)^3 - (3^{18}2^3x)^2 = 1728(-3^3z)^{11}$. Hence, Corollary 1.3 provides a new analytic construction of solutions for the Diophantine equation $x^p + y^q = z^r$ in the more difficult case $(p, q, r) = (2, 3, 11)$, which goes beyond the known ten solutions in the Fermat-Catalan Conjecture and gives an answer to Problem 4. Theorem 1.1 and Theorem 1.2 give an answer to a problem dating back to the works of Klein (see the end of section four).

In number theory and arithmetical algebraic geometry, a central problem is the modularity of algebraic varieties. Our example of exotic structure on the equation of E_8 -singularity shows that the algebraic structure of a variety does not completely determine its arithmetical (modular) structure. This leads to the study of the relationships between algebraic property and arithmetical (modular) property on an algebraic variety. Note that the Klein quartic curve has two kinds of modular parametrizations with the same level and the same symmetry group. One is the modular curve $X(7)$ of level 7 with symmetry group $\Gamma/\Gamma(7) \cong \text{PSL}(2, 7)$. The other is the Shimura curve $\mathbb{H}/\Gamma(\mathfrak{o}, \mathfrak{p})$ for a certain quaternion algebra \mathfrak{o} over a cubic field $\mathbb{Q}(\cos \frac{2\pi}{7})$ and $N(\mathfrak{p}) = 7$, whose symmetry group is $\Gamma(\mathfrak{o})/\Gamma(\mathfrak{o}, \mathfrak{p}) \cong \text{PSL}(2, 7)$ (see [Sh]). In contrast with it, our example has different levels and different symmetry groups.

In fact, G is a finite subgroup of $\text{SL}(6, \mathbb{C})$ (see [L]) and \mathbb{C}^6/G is a six-dimensional quotient singularity. Theorem 1.1 and Theorem 1.2 imply that each of the following nine triples $(\Phi_{12}, \Phi_{18}, \Phi_{20})$, $(\Phi_{12}, \Phi_{20}, \Phi_{30})$, $(\Phi_{12}, \Phi_{20}, \Phi_{42})$, $(\Phi_{12}, \Phi_{18}, \Phi_{32})$, $(\Phi_{12}, \Phi_{30}, \Phi_{32})$, $(\Phi_{12}, \Phi_{32}, \Phi_{42})$, $(\Phi_{12}, \Phi_{18}, \Phi_{44})$, $(\Phi_{12}, \Phi_{30}, \Phi_{44})$ and $(\Phi_{12}, \Phi_{42}, \Phi_{44})$ gives a map from \mathbb{C}^6/G to \mathbb{C}^3 with the corresponding syzygy which relates the three polynomials in six variables when evaluating at theta constants of order thirteen.

This paper consists of four sections. In section two, we give the standard structure on the E_8 -singularity. In section three, we obtain the invariant theory for $\text{PSL}(2, 13)$. In particular, we construct the senary quadratic forms \mathbb{A}_j ($0 \leq j \leq 6$), the senary cubic forms \mathbb{D}_j ($j = 0, 1, \dots, 12, \infty$) and the senary sextic forms \mathbb{G}_j ($0 \leq j \leq 12$). From \mathbb{A}_j we construct the Jacobian equation of degree fourteen. From \mathbb{D}_j and \mathbb{G}_j we construct the exotic equation of degree fourteen. It should be pointed out that the appearance of exotic equation is a new phenomenon which does not appear in the invariant theory for $\text{PSL}(2, 5)$, $\text{PSL}(2, 7)$ and $\text{PSL}(2, 11)$ (see [K], [K1], [K2], [K3], [K4], [KF1] and [KF2]). Combining Jacobian equation with exotic equation, we obtain the polynomials Φ_{12} , Φ_{18} , Φ_{20} and Φ_{30} which are invariant under the action of $\text{PSL}(2, 13)$. Together with theta constants of order thirteen, we give the exotic structure on the E_8 -singularity and prove Theorem 1.1. In section four, we obtain the invariant polynomials Φ_{32} , Φ_{42} , and Φ_{44} . Together with theta constants of order thirteen, we give modular parametrizations for the exceptional singularities Q_{18} , E_{20} and $x^7 + x^2y^3 + z^2 = 0$ and prove Theorem 1.2.

2. Standard structure on the E_8 -singularity: the icosahedron

In this section, we will give the standard structure on the equation of E_8 -singularity: the classical icosahedral equation with icosahedral symmetry, which has a modular interpretation in terms of theta constants of order five.

The most famous source on icosahedral symmetry is the celebrated book [K] of Felix Klein on the icosahedron and the solution of quintic equations which appeared for the first time in 1884. Its main objective is to show that the solution of general quintic equations can be reduced to that of locally inverting certain quotients of actions of the icosahedral group $G \cong A_5$ on spaces related to the usual geometric realization of the regular icosahedron in three-dimensional space. In particular, Klein (see [K]) gave a parametric solution of the E_8 -singularity using the icosahedron by homogeneous polynomials T , H , f in two variables of degrees 30, 20, 12 with integral coefficients. Here,

$$\begin{aligned} f &= z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}), \\ H &= \frac{1}{121} \left| \begin{array}{cc} \frac{\partial^2 f}{\partial z_1^2} & \frac{\partial^2 f}{\partial z_1 \partial z_2} \\ \frac{\partial^2 f}{\partial z_2 \partial z_1} & \frac{\partial^2 f}{\partial z_2^2} \end{array} \right| = -(z_1^{20} + z_2^{20}) + 228(z_1^{15} z_2^5 - z_1^5 z_2^{15}) - 494 z_1^{10} z_2^{10}, \\ T &= -\frac{1}{20} \left| \begin{array}{cc} \frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} \\ \frac{\partial H}{\partial z_1} & \frac{\partial H}{\partial z_2} \end{array} \right| = (z_1^{30} + z_2^{30}) + 522(z_1^{25} z_2^5 - z_1^5 z_2^{25}) - 10005(z_1^{20} z_2^{10} + z_1^{10} z_2^{20}). \end{aligned}$$

They satisfy the famous icosahedral equation

$$T^2 + H^3 = 1728 f^5. \quad (2.1)$$

In fact, f , H and T are invariant polynomials under the action of the icosahedral group. Essentially the same relation had been found a few years earlier by Schwarz (see [Sch]), who considered three polynomials φ_{12} , φ_{20} and φ_{30} whose roots correspond to the vertices, the midpoints of the faces and the midpoints of the edges of an icosahedron inscribed in the Riemann sphere. He obtained the identity $\varphi_{20}^3 - 1728 \varphi_{12}^5 = \varphi_{30}^2$. Thus we see that from the very beginning there was a close relation between the E_8 -singularity and the icosahedron. Moreover, the icosahedral equation (2.1) can be interpreted in terms of modular forms which was also known by Klein (see [KF1], p. 631). Let $x_1(z) = \eta(z)a(z)$ and $x_2(z) = \eta(z)b(z)$, where

$$a(z) = e^{-\frac{3\pi i}{10}\theta} \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, 5z), \quad b(z) = e^{-\frac{\pi i}{10}\theta} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, 5z)$$

are theta constants of order five and $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi i z}$ is the Dedekind eta function which are all defined in the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Then

$$\begin{cases} f(x_1(z), x_2(z)) = -\Delta(z), \\ H(x_1(z), x_2(z)) = -\eta(z)^8 \Delta(z) E_4(z), \\ T(x_1(z), x_2(z)) = \Delta(z)^2 E_6(z), \end{cases}$$

where $E_4(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n)=1} \frac{1}{(mz+n)^4}$ and $E_6(z) := \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n)=1} \frac{1}{(mz+n)^6}$ are Eisenstein series of weight 4 and 6, and $\Delta(z) = \eta(z)^{24}$ is the discriminant. The relations

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = \frac{H(x_1(z), x_2(z))^3}{f(x_1(z), x_2(z))^5}, \quad j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = -\frac{T(x_1(z), x_2(z))^2}{f(x_1(z), x_2(z))^5}$$

give the icosahedral equation (2.1) in terms of theta constants of order five.

3. Exotic structure on the E_8 -singularity: $\mathrm{PSL}(2, 13)$

In this section, we will give the exotic structure on the equation of E_8 -singularity: the symmetry group is the simple group $\mathrm{PSL}(2, 13)$ and the equation has a modular interpretation in terms of theta constants of order thirteen.

At first, we will study the six-dimensional representation of the finite simple group $\mathrm{PSL}(2, 13)$ of order 1092, which acts on the five-dimensional projective space $\mathbb{P}^5 = \{(z_1, z_2, z_3, z_4, z_5, z_6) : z_i \in \mathbb{C} \ (i = 1, 2, 3, 4, 5, 6)\}$. This representation is defined over the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{13}})$. Put

$$S = -\frac{1}{\sqrt{13}} \begin{pmatrix} \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 \\ \zeta^{10} - \zeta^3 & \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 \\ \zeta^4 - \zeta^9 & \zeta^{12} - \zeta & \zeta^{10} - \zeta^3 & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} \\ \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 \\ \zeta^2 - \zeta^{11} & \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^3 - \zeta^{10} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} \\ \zeta^6 - \zeta^7 & \zeta^5 - \zeta^8 & \zeta^2 - \zeta^{11} & \zeta^9 - \zeta^4 & \zeta - \zeta^{12} & \zeta^3 - \zeta^{10} \end{pmatrix} \quad (3.1)$$

and

$$T = \mathrm{diag}(\zeta^7, \zeta^{11}, \zeta^8, \zeta^6, \zeta^2, \zeta^5), \quad (3.2)$$

where $\zeta = \exp(2\pi i/13)$. We have

$$S^2 = T^{13} = (ST)^3 = 1. \quad (3.3)$$

Let $G = \langle S, T \rangle$, then $G \cong \mathrm{PSL}(2, 13)$ (see [Y1], Theorem 3.1).

Put $\theta_1 = \zeta + \zeta^3 + \zeta^9$, $\theta_2 = \zeta^2 + \zeta^6 + \zeta^5$, $\theta_3 = \zeta^4 + \zeta^{12} + \zeta^{10}$, and $\theta_4 = \zeta^8 + \zeta^{11} + \zeta^7$. We find that

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 + \theta_4 = -1, \\ \theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4 = 2, \\ \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4 = 4, \\ \theta_1\theta_2\theta_3\theta_4 = 3. \end{cases}$$

Hence, $\theta_1, \theta_2, \theta_3$ and θ_4 satisfy the quartic equation $z^4 + z^3 + 2z^2 - 4z + 3 = 0$, which can be decomposed as two quadratic equations

$$\left(z^2 + \frac{1 + \sqrt{13}}{2}z + \frac{5 + \sqrt{13}}{2}\right) \left(z^2 + \frac{1 - \sqrt{13}}{2}z + \frac{5 - \sqrt{13}}{2}\right) = 0$$

over the real quadratic field $\mathbb{Q}(\sqrt{13})$. Therefore, the four roots are given as follows:

$$\begin{cases} \theta_1 = \frac{1}{4} \left(-1 + \sqrt{13} + \sqrt{-26 + 6\sqrt{13}} \right), \\ \theta_2 = \frac{1}{4} \left(-1 - \sqrt{13} + \sqrt{-26 - 6\sqrt{13}} \right), \\ \theta_3 = \frac{1}{4} \left(-1 + \sqrt{13} - \sqrt{-26 + 6\sqrt{13}} \right), \\ \theta_4 = \frac{1}{4} \left(-1 - \sqrt{13} - \sqrt{-26 - 6\sqrt{13}} \right). \end{cases}$$

Moreover, we find that

$$\begin{cases} \theta_1 + \theta_3 + \theta_2 + \theta_4 = -1, \\ \theta_1 + \theta_3 - \theta_2 - \theta_4 = \sqrt{13}, \\ \theta_1 - \theta_3 - \theta_2 + \theta_4 = -\sqrt{-13 + 2\sqrt{13}}, \\ \theta_1 - \theta_3 + \theta_2 - \theta_4 = \sqrt{-13 - 2\sqrt{13}}. \end{cases}$$

Let us study the action of ST^ν on \mathbb{P}^5 , where $\nu = 0, 1, \dots, 12$. Put

$$\alpha = \zeta + \zeta^{12} - \zeta^5 - \zeta^8, \quad \beta = \zeta^3 + \zeta^{10} - \zeta^2 - \zeta^{11}, \quad \gamma = \zeta^9 + \zeta^4 - \zeta^6 - \zeta^7.$$

We find that

$$\begin{aligned} & 13ST^\nu(z_1) \cdot ST^\nu(z_4) \\ &= \beta z_1 z_4 + \gamma z_2 z_5 + \alpha z_3 z_6 + \\ & \quad + \gamma \zeta^\nu z_1^2 + \alpha \zeta^{9\nu} z_2^2 + \beta \zeta^{3\nu} z_3^2 - \gamma \zeta^{12\nu} z_4^2 - \alpha \zeta^{4\nu} z_5^2 - \beta \zeta^{10\nu} z_6^2 + \\ & \quad + (\alpha - \beta) \zeta^{5\nu} z_1 z_2 + (\beta - \gamma) \zeta^{6\nu} z_2 z_3 + (\gamma - \alpha) \zeta^{2\nu} z_1 z_3 + \\ & \quad + (\beta - \alpha) \zeta^{8\nu} z_4 z_5 + (\gamma - \beta) \zeta^{7\nu} z_5 z_6 + (\alpha - \gamma) \zeta^{11\nu} z_4 z_6 + \\ & \quad - (\alpha + \beta) \zeta^\nu z_3 z_4 - (\beta + \gamma) \zeta^{9\nu} z_1 z_5 - (\gamma + \alpha) \zeta^{3\nu} z_2 z_6 + \\ & \quad - (\alpha + \beta) \zeta^{12\nu} z_1 z_6 - (\beta + \gamma) \zeta^{4\nu} z_2 z_4 - (\gamma + \alpha) \zeta^{10\nu} z_3 z_5. \\ & 13ST^\nu(z_2) \cdot ST^\nu(z_5) \\ &= \gamma z_1 z_4 + \alpha z_2 z_5 + \beta z_3 z_6 + \\ & \quad + \alpha \zeta^\nu z_1^2 + \beta \zeta^{9\nu} z_2^2 + \gamma \zeta^{3\nu} z_3^2 - \alpha \zeta^{12\nu} z_4^2 - \beta \zeta^{4\nu} z_5^2 - \gamma \zeta^{10\nu} z_6^2 + \\ & \quad + (\beta - \gamma) \zeta^{5\nu} z_1 z_2 + (\gamma - \alpha) \zeta^{6\nu} z_2 z_3 + (\alpha - \beta) \zeta^{2\nu} z_1 z_3 + \\ & \quad + (\gamma - \beta) \zeta^{8\nu} z_4 z_5 + (\alpha - \gamma) \zeta^{7\nu} z_5 z_6 + (\beta - \alpha) \zeta^{11\nu} z_4 z_6 + \\ & \quad - (\beta + \gamma) \zeta^\nu z_3 z_4 - (\gamma + \alpha) \zeta^{9\nu} z_1 z_5 - (\alpha + \beta) \zeta^{3\nu} z_2 z_6 + \\ & \quad - (\beta + \gamma) \zeta^{12\nu} z_1 z_6 - (\gamma + \alpha) \zeta^{4\nu} z_2 z_4 - (\alpha + \beta) \zeta^{10\nu} z_3 z_5. \end{aligned}$$

$$\begin{aligned}
& 13ST^\nu(z_3) \cdot ST^\nu(z_6) \\
&= \alpha z_1 z_4 + \beta z_2 z_5 + \gamma z_3 z_6 + \\
& \quad + \beta \zeta^\nu z_1^2 + \gamma \zeta^{9\nu} z_2^2 + \alpha \zeta^{3\nu} z_3^2 - \beta \zeta^{12\nu} z_4^2 - \gamma \zeta^{4\nu} z_5^2 - \alpha \zeta^{10\nu} z_6^2 + \\
& \quad + (\gamma - \alpha) \zeta^{5\nu} z_1 z_2 + (\alpha - \beta) \zeta^{6\nu} z_2 z_3 + (\beta - \gamma) \zeta^{2\nu} z_1 z_3 + \\
& \quad + (\alpha - \gamma) \zeta^{8\nu} z_4 z_5 + (\beta - \alpha) \zeta^{7\nu} z_5 z_6 + (\gamma - \beta) \zeta^{11\nu} z_4 z_6 + \\
& \quad - (\gamma + \alpha) \zeta^\nu z_3 z_4 - (\alpha + \beta) \zeta^{9\nu} z_1 z_5 - (\beta + \gamma) \zeta^{3\nu} z_2 z_6 + \\
& \quad - (\gamma + \alpha) \zeta^{12\nu} z_1 z_6 - (\alpha + \beta) \zeta^{4\nu} z_2 z_4 - (\beta + \gamma) \zeta^{10\nu} z_3 z_5.
\end{aligned}$$

Note that $\alpha + \beta + \gamma = \sqrt{13}$, we find that

$$\begin{aligned}
& \sqrt{13} [ST^\nu(z_1) \cdot ST^\nu(z_4) + ST^\nu(z_2) \cdot ST^\nu(z_5) + ST^\nu(z_3) \cdot ST^\nu(z_6)] \\
&= (z_1 z_4 + z_2 z_5 + z_3 z_6) + (\zeta^\nu z_1^2 + \zeta^{9\nu} z_2^2 + \zeta^{3\nu} z_3^2) - (\zeta^{12\nu} z_4^2 + \zeta^{4\nu} z_5^2 + \zeta^{10\nu} z_6^2) + \\
& \quad - 2(\zeta^\nu z_3 z_4 + \zeta^{9\nu} z_1 z_5 + \zeta^{3\nu} z_2 z_6) - 2(\zeta^{12\nu} z_1 z_6 + \zeta^{4\nu} z_2 z_4 + \zeta^{10\nu} z_3 z_5).
\end{aligned}$$

Let

$$\varphi_\infty(z_1, z_2, z_3, z_4, z_5, z_6) = \sqrt{13}(z_1 z_4 + z_2 z_5 + z_3 z_6) \quad (3.4)$$

and

$$\varphi_\nu(z_1, z_2, z_3, z_4, z_5, z_6) = \varphi_\infty(ST^\nu(z_1, z_2, z_3, z_4, z_5, z_6)) \quad (3.5)$$

for $\nu = 0, 1, \dots, 12$. Then

$$\begin{aligned}
\varphi_\nu = & (z_1 z_4 + z_2 z_5 + z_3 z_6) + \zeta^\nu (z_1^2 - 2z_3 z_4) + \zeta^{4\nu} (-z_5^2 - 2z_2 z_4) + \\
& + \zeta^{9\nu} (z_2^2 - 2z_1 z_5) + \zeta^{3\nu} (z_3^2 - 2z_2 z_6) + \zeta^{12\nu} (-z_4^2 - 2z_1 z_6) + \zeta^{10\nu} (-z_6^2 - 2z_3 z_5).
\end{aligned} \quad (3.6)$$

This leads us to define the following senary quadratic forms (quadratic forms in six variables):

$$\begin{cases} \mathbb{A}_0 = z_1 z_4 + z_2 z_5 + z_3 z_6, \\ \mathbb{A}_1 = z_1^2 - 2z_3 z_4, \\ \mathbb{A}_2 = -z_5^2 - 2z_2 z_4, \\ \mathbb{A}_3 = z_2^2 - 2z_1 z_5, \\ \mathbb{A}_4 = z_3^2 - 2z_2 z_6, \\ \mathbb{A}_5 = -z_4^2 - 2z_1 z_6, \\ \mathbb{A}_6 = -z_6^2 - 2z_3 z_5. \end{cases} \quad (3.7)$$

Hence,

$$\sqrt{13}ST^\nu(\mathbb{A}_0) = \mathbb{A}_0 + \zeta^\nu \mathbb{A}_1 + \zeta^{4\nu} \mathbb{A}_2 + \zeta^{9\nu} \mathbb{A}_3 + \zeta^{3\nu} \mathbb{A}_4 + \zeta^{12\nu} \mathbb{A}_5 + \zeta^{10\nu} \mathbb{A}_6. \quad (3.8)$$

Let $H := Q^5 P^2 \cdot P^2 Q^6 P^8 \cdot Q^5 P^2 \cdot P^3 Q$ where $P = ST^{-1}S$ and $Q = ST^3$. Then (see [Y2], p.27)

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.9)$$

Note that $H^6 = 1$ and $H^{-1}TH = -T^4$. Thus, $\langle H, T \rangle \cong \mathbb{Z}_{13} \rtimes \mathbb{Z}_6$. Hence, it is a maximal subgroup of order 78 of G with index 14 (see [CC]). We find that φ_∞^2 is invariant under the action of the maximal subgroup $\langle H, T \rangle$. Note that

$$\varphi_\infty = \sqrt{13}\mathbb{A}_0, \quad \varphi_\nu = \mathbb{A}_0 + \zeta^\nu \mathbb{A}_1 + \zeta^{4\nu} \mathbb{A}_2 + \zeta^{9\nu} \mathbb{A}_3 + \zeta^{3\nu} \mathbb{A}_4 + \zeta^{12\nu} \mathbb{A}_5 + \zeta^{10\nu} \mathbb{A}_6$$

for $\nu = 0, 1, \dots, 12$. Let $w = \varphi^2$, $w_\infty = \varphi_\infty^2$ and $w_\nu = \varphi_\nu^2$. Then w_∞, w_ν for $\nu = 0, \dots, 12$ form an algebraic equation of degree fourteen, which is just the Jacobian equation of degree fourteen, whose roots are these w_ν and w_∞ :

$$w^{14} + a_1 w^{13} + \dots + a_{13} w + a_{14} = 0.$$

On the other hand, we have

$$\begin{aligned} & -13\sqrt{13}ST^\nu(z_1) \cdot ST^\nu(z_2) \cdot ST^\nu(z_3) \\ = & -r_4(\zeta^{8\nu} z_1^3 + \zeta^{7\nu} z_2^3 + \zeta^{11\nu} z_3^3) - r_2(\zeta^{5\nu} z_4^3 + \zeta^{6\nu} z_5^3 + \zeta^{2\nu} z_6^3) \\ & - r_3(\zeta^{12\nu} z_1^2 z_2 + \zeta^{4\nu} z_2^2 z_3 + \zeta^{10\nu} z_3^2 z_1) - r_1(\zeta^\nu z_4^2 z_5 + \zeta^{9\nu} z_5^2 z_6 + \zeta^{3\nu} z_6^2 z_4) \\ & + 2r_1(\zeta^{3\nu} z_1 z_2^2 + \zeta^\nu z_2 z_3^2 + \zeta^{9\nu} z_3 z_1^2) - 2r_3(\zeta^{10\nu} z_4 z_5^2 + \zeta^{12\nu} z_5 z_6^2 + \zeta^{4\nu} z_6 z_4^2) \\ & + 2r_4(\zeta^{7\nu} z_1^2 z_4 + \zeta^{11\nu} z_2^2 z_5 + \zeta^{8\nu} z_3^2 z_6) - 2r_2(\zeta^{6\nu} z_1 z_4^2 + \zeta^{2\nu} z_2 z_5^2 + \zeta^{5\nu} z_3 z_6^2) + \\ & + r_1(\zeta^{3\nu} z_1^2 z_5 + \zeta^\nu z_2^2 z_6 + \zeta^{9\nu} z_3^2 z_4) + r_3(\zeta^{10\nu} z_2 z_4^2 + \zeta^{12\nu} z_3 z_5^2 + \zeta^{4\nu} z_1 z_6^2) + \\ & + r_2(\zeta^{6\nu} z_1^2 z_6 + \zeta^{2\nu} z_2^2 z_4 + \zeta^{5\nu} z_3^2 z_5) + r_4(\zeta^{7\nu} z_3 z_4^2 + \zeta^{11\nu} z_1 z_5^2 + \zeta^{8\nu} z_2 z_6^2) + \\ & + r_0 z_1 z_2 z_3 + r_\infty z_4 z_5 z_6 + \\ & - r_4(\zeta^{11\nu} z_1 z_2 z_4 + \zeta^{8\nu} z_2 z_3 z_5 + \zeta^{7\nu} z_1 z_3 z_6) + \\ & + r_2(\zeta^{2\nu} z_1 z_4 z_5 + \zeta^{5\nu} z_2 z_5 z_6 + \zeta^{6\nu} z_3 z_4 z_6) + \\ & - 3r_4(\zeta^{7\nu} z_1 z_2 z_5 + \zeta^{11\nu} z_2 z_3 z_6 + \zeta^{8\nu} z_1 z_3 z_4) + \\ & + 3r_2(\zeta^{6\nu} z_2 z_4 z_5 + \zeta^{2\nu} z_3 z_5 z_6 + \zeta^{5\nu} z_1 z_4 z_6) + \\ & - r_3(\zeta^{10\nu} z_1 z_2 z_6 + \zeta^{4\nu} z_1 z_3 z_5 + \zeta^{12\nu} z_2 z_3 z_4) + \\ & + r_1(\zeta^{3\nu} z_3 z_4 z_5 + \zeta^{9\nu} z_2 z_4 z_6 + \zeta^\nu z_1 z_5 z_6), \end{aligned}$$

where

$$\begin{aligned} r_0 &= 2(\theta_1 - \theta_3) - 3(\theta_2 - \theta_4), & r_\infty &= 2(\theta_4 - \theta_2) - 3(\theta_1 - \theta_3), \\ r_1 &= \sqrt{-13 - 2\sqrt{13}}, & r_2 &= \sqrt{\frac{-13 + 3\sqrt{13}}{2}}, \\ r_3 &= \sqrt{-13 + 2\sqrt{13}}, & r_4 &= \sqrt{\frac{-13 - 3\sqrt{13}}{2}}. \end{aligned}$$

This leads us to define the following senary cubic forms (cubic forms in six variables):

$$\left\{ \begin{array}{l} \mathbb{D}_0 = z_1 z_2 z_3, \\ \mathbb{D}_1 = 2z_2 z_3^2 + z_2^2 z_6 - z_4^2 z_5 + z_1 z_5 z_6, \\ \mathbb{D}_2 = -z_6^3 + z_2^2 z_4 - 2z_2 z_5^2 + z_1 z_4 z_5 + 3z_3 z_5 z_6, \\ \mathbb{D}_3 = 2z_1 z_2^2 + z_1^2 z_5 - z_4 z_6^2 + z_3 z_4 z_5, \\ \mathbb{D}_4 = -z_2^2 z_3 + z_1 z_6^2 - 2z_4^2 z_6 - z_1 z_3 z_5, \\ \mathbb{D}_5 = -z_4^3 + z_3^2 z_5 - 2z_3 z_6^2 + z_2 z_5 z_6 + 3z_1 z_4 z_6, \\ \mathbb{D}_6 = -z_5^3 + z_1^2 z_6 - 2z_1 z_4^2 + z_3 z_4 z_6 + 3z_2 z_4 z_5, \\ \mathbb{D}_7 = -z_2^3 + z_3 z_4^2 - z_1 z_3 z_6 - 3z_1 z_2 z_5 + 2z_1^2 z_4, \\ \mathbb{D}_8 = -z_1^3 + z_2 z_6^2 - z_2 z_3 z_5 - 3z_1 z_3 z_4 + 2z_3^2 z_6, \\ \mathbb{D}_9 = 2z_1^2 z_3 + z_3^2 z_4 - z_5^2 z_6 + z_2 z_4 z_6, \\ \mathbb{D}_{10} = -z_1 z_3^2 + z_2 z_4^2 - 2z_4 z_5^2 - z_1 z_2 z_6, \\ \mathbb{D}_{11} = -z_3^3 + z_1 z_5^2 - z_1 z_2 z_4 - 3z_2 z_3 z_6 + 2z_2^2 z_5, \\ \mathbb{D}_{12} = -z_1^2 z_2 + z_3 z_5^2 - 2z_5 z_6^2 - z_2 z_3 z_4, \\ \mathbb{D}_\infty = z_4 z_5 z_6. \end{array} \right. \quad (3.10)$$

Then

$$\begin{aligned} & -13\sqrt{13}ST^\nu(\mathbb{D}_0) \\ & = r_0 \mathbb{D}_0 + r_1 \zeta^\nu \mathbb{D}_1 + r_2 \zeta^{2\nu} \mathbb{D}_2 + r_1 \zeta^{3\nu} \mathbb{D}_3 + r_3 \zeta^{4\nu} \mathbb{D}_4 + r_2 \zeta^{5\nu} \mathbb{D}_5 + r_2 \zeta^{6\nu} \mathbb{D}_6 + \\ & \quad + r_4 \zeta^{7\nu} \mathbb{D}_7 + r_4 \zeta^{8\nu} \mathbb{D}_8 + r_1 \zeta^{9\nu} \mathbb{D}_9 + r_3 \zeta^{10\nu} \mathbb{D}_{10} + r_4 \zeta^{11\nu} \mathbb{D}_{11} + r_3 \zeta^{12\nu} \mathbb{D}_{12} + r_\infty \mathbb{D}_\infty. \\ & -13\sqrt{13}ST^\nu(\mathbb{D}_\infty) \\ & = r_\infty \mathbb{D}_0 - r_3 \zeta^\nu \mathbb{D}_1 - r_4 \zeta^{2\nu} \mathbb{D}_2 - r_3 \zeta^{3\nu} \mathbb{D}_3 + r_1 \zeta^{4\nu} \mathbb{D}_4 - r_4 \zeta^{5\nu} \mathbb{D}_5 - r_4 \zeta^{6\nu} \mathbb{D}_6 + \\ & \quad + r_2 \zeta^{7\nu} \mathbb{D}_7 + r_2 \zeta^{8\nu} \mathbb{D}_8 - r_3 \zeta^{9\nu} \mathbb{D}_9 + r_1 \zeta^{10\nu} \mathbb{D}_{10} + r_2 \zeta^{11\nu} \mathbb{D}_{11} + r_1 \zeta^{12\nu} \mathbb{D}_{12} - r_0 \mathbb{D}_\infty. \end{aligned}$$

Let

$$\delta_\infty(z_1, z_2, z_3, z_4, z_5, z_6) = 13^2(z_1^2 z_2^2 z_3^2 + z_4^2 z_5^2 z_6^2) \quad (3.11)$$

and

$$\delta_\nu(z_1, z_2, z_3, z_4, z_5, z_6) = \delta_\infty(ST^\nu(z_1, z_2, z_3, z_4, z_5, z_6)) \quad (3.12)$$

for $\nu = 0, 1, \dots, 12$. Then

$$\delta_\nu = 13^2 ST^\nu(\mathbb{G}_0) = -13\mathbb{G}_0 + \zeta^\nu \mathbb{G}_1 + \zeta^{2\nu} \mathbb{G}_2 + \dots + \zeta^{12\nu} \mathbb{G}_{12}, \quad (3.13)$$

where the senary sextic forms (i.e., sextic forms in six variables) are given as follows:

$$\left\{ \begin{array}{l} \mathbb{G}_0 = \mathbb{D}_0^2 + \mathbb{D}_\infty^2, \\ \mathbb{G}_1 = -\mathbb{D}_7^2 + 2\mathbb{D}_0\mathbb{D}_1 + 10\mathbb{D}_\infty\mathbb{D}_1 + 2\mathbb{D}_2\mathbb{D}_{12} - 2\mathbb{D}_3\mathbb{D}_{11} - 4\mathbb{D}_4\mathbb{D}_{10} - 2\mathbb{D}_9\mathbb{D}_5, \\ \mathbb{G}_2 = -2\mathbb{D}_1^2 - 4\mathbb{D}_0\mathbb{D}_2 + 6\mathbb{D}_\infty\mathbb{D}_2 - 2\mathbb{D}_4\mathbb{D}_{11} + 2\mathbb{D}_5\mathbb{D}_{10} - 2\mathbb{D}_6\mathbb{D}_9 - 2\mathbb{D}_7\mathbb{D}_8, \\ \mathbb{G}_3 = -\mathbb{D}_8^2 + 2\mathbb{D}_0\mathbb{D}_3 + 10\mathbb{D}_\infty\mathbb{D}_3 + 2\mathbb{D}_6\mathbb{D}_{10} - 2\mathbb{D}_9\mathbb{D}_7 - 4\mathbb{D}_{12}\mathbb{D}_4 - 2\mathbb{D}_1\mathbb{D}_2, \\ \mathbb{G}_4 = -\mathbb{D}_2^2 + 10\mathbb{D}_0\mathbb{D}_4 - 2\mathbb{D}_\infty\mathbb{D}_4 + 2\mathbb{D}_5\mathbb{D}_{12} - 2\mathbb{D}_9\mathbb{D}_8 - 4\mathbb{D}_1\mathbb{D}_3 - 2\mathbb{D}_{10}\mathbb{D}_7, \\ \mathbb{G}_5 = -2\mathbb{D}_9^2 - 4\mathbb{D}_0\mathbb{D}_5 + 6\mathbb{D}_\infty\mathbb{D}_5 - 2\mathbb{D}_{10}\mathbb{D}_8 + 2\mathbb{D}_6\mathbb{D}_{12} - 2\mathbb{D}_2\mathbb{D}_3 - 2\mathbb{D}_{11}\mathbb{D}_7, \\ \mathbb{G}_6 = -2\mathbb{D}_3^2 - 4\mathbb{D}_0\mathbb{D}_6 + 6\mathbb{D}_\infty\mathbb{D}_6 - 2\mathbb{D}_{12}\mathbb{D}_7 + 2\mathbb{D}_2\mathbb{D}_4 - 2\mathbb{D}_5\mathbb{D}_1 - 2\mathbb{D}_8\mathbb{D}_{11}, \\ \mathbb{G}_7 = -2\mathbb{D}_{10}^2 + 6\mathbb{D}_0\mathbb{D}_7 + 4\mathbb{D}_\infty\mathbb{D}_7 - 2\mathbb{D}_1\mathbb{D}_6 - 2\mathbb{D}_2\mathbb{D}_5 - 2\mathbb{D}_8\mathbb{D}_{12} - 2\mathbb{D}_9\mathbb{D}_{11}, \\ \mathbb{G}_8 = -2\mathbb{D}_4^2 + 6\mathbb{D}_0\mathbb{D}_8 + 4\mathbb{D}_\infty\mathbb{D}_8 - 2\mathbb{D}_3\mathbb{D}_5 - 2\mathbb{D}_6\mathbb{D}_2 - 2\mathbb{D}_{11}\mathbb{D}_{10} - 2\mathbb{D}_1\mathbb{D}_7, \\ \mathbb{G}_9 = -\mathbb{D}_{11}^2 + 2\mathbb{D}_0\mathbb{D}_9 + 10\mathbb{D}_\infty\mathbb{D}_9 + 2\mathbb{D}_5\mathbb{D}_4 - 2\mathbb{D}_1\mathbb{D}_8 - 4\mathbb{D}_{10}\mathbb{D}_{12} - 2\mathbb{D}_3\mathbb{D}_6, \\ \mathbb{G}_{10} = -\mathbb{D}_5^2 + 10\mathbb{D}_0\mathbb{D}_{10} - 2\mathbb{D}_\infty\mathbb{D}_{10} + 2\mathbb{D}_6\mathbb{D}_4 - 2\mathbb{D}_3\mathbb{D}_7 - 4\mathbb{D}_9\mathbb{D}_1 - 2\mathbb{D}_{12}\mathbb{D}_{11}, \\ \mathbb{G}_{11} = -2\mathbb{D}_{12}^2 + 6\mathbb{D}_0\mathbb{D}_{11} + 4\mathbb{D}_\infty\mathbb{D}_{11} - 2\mathbb{D}_9\mathbb{D}_2 - 2\mathbb{D}_5\mathbb{D}_6 - 2\mathbb{D}_7\mathbb{D}_4 - 2\mathbb{D}_3\mathbb{D}_8, \\ \mathbb{G}_{12} = -\mathbb{D}_6^2 + 10\mathbb{D}_0\mathbb{D}_{12} - 2\mathbb{D}_\infty\mathbb{D}_{12} + 2\mathbb{D}_2\mathbb{D}_{10} - 2\mathbb{D}_1\mathbb{D}_{11} - 4\mathbb{D}_3\mathbb{D}_9 - 2\mathbb{D}_4\mathbb{D}_8. \end{array} \right. \quad (3.14)$$

We have that \mathbb{G}_0 is invariant under the action of $\langle H, T \rangle$, a maximal subgroup of order 78 of G with index 14. Note that $\delta_\infty, \delta_\nu$ for $\nu = 0, \dots, 12$ form an algebraic equation of degree fourteen. However, we have $\delta_\infty + \sum_{\nu=0}^{12} \delta_\nu = 0$. Hence, it is not the Jacobian equation of degree fourteen! We call it the exotic equation of degree fourteen.

Recall that the theta functions with characteristic $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2$ is defined by the following series which converges uniformly and absolutely on compact subsets of $\mathbb{C} \times \mathbb{H}$ (see [FK], p. 73):

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}} \exp \left\{ 2\pi i \left[\frac{1}{2} \left(n + \frac{\epsilon}{2} \right)^2 \tau + \left(n + \frac{\epsilon}{2} \right) \left(z + \frac{\epsilon'}{2} \right) \right] \right\}.$$

The modified theta constants (see [FK], p. 215) $\varphi_l(\tau) := \theta[\chi_l](0, k\tau)$, where the characteristic $\chi_l = \begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix}$, $l = 0, \dots, \frac{k-3}{2}$, for odd k and $\chi_l = \begin{bmatrix} \frac{2l}{k} \\ 0 \end{bmatrix}$, $l = 0, \dots, \frac{k}{2}$, for even k . We have the following:

Proposition 3.1. (see [FK], p. 236). *For each odd integer $k \geq 5$, the map $\Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \dots, \varphi_{\frac{k-5}{2}}(\tau), \varphi_{\frac{k-3}{2}}(\tau))$ from $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ to $\mathbb{C}^{\frac{k-1}{2}}$, defines a holomorphic mapping from $\overline{\mathbb{H}}/\Gamma(k)$ into $\mathbb{CP}^{\frac{k-3}{2}}$.*

In our case, the map $\Phi : \tau \mapsto (\varphi_0(\tau), \varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau), \varphi_4(\tau), \varphi_5(\tau))$ gives a holomorphic mapping from the modular curve $X(13) = \mathbb{H}/\Gamma(13)$ into \mathbb{CP}^5 , which corresponds to our six-dimensional representation, i.e., up to the constants, z_1, \dots, z_6 are just modular forms $\varphi_0(\tau), \dots, \varphi_5(\tau)$. Let

$$\left\{ \begin{array}{l} a_1(z) := e^{-\frac{11\pi i}{26}} \theta \begin{bmatrix} \frac{11}{13} \\ 1 \end{bmatrix} (0, 13z) = q^{\frac{121}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2 + 11n)}, \\ a_2(z) := e^{-\frac{7\pi i}{26}} \theta \begin{bmatrix} \frac{7}{13} \\ 1 \end{bmatrix} (0, 13z) = q^{\frac{49}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2 + 7n)}, \\ a_3(z) := e^{-\frac{5\pi i}{26}} \theta \begin{bmatrix} \frac{5}{13} \\ 1 \end{bmatrix} (0, 13z) = q^{\frac{25}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2 + 5n)}, \\ a_4(z) := -e^{-\frac{3\pi i}{26}} \theta \begin{bmatrix} \frac{3}{13} \\ 1 \end{bmatrix} (0, 13z) = -q^{\frac{9}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2 + 3n)}, \\ a_5(z) := e^{-\frac{9\pi i}{26}} \theta \begin{bmatrix} \frac{9}{13} \\ 1 \end{bmatrix} (0, 13z) = q^{\frac{81}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2 + 9n)}, \\ a_6(z) := e^{-\frac{\pi i}{26}} \theta \begin{bmatrix} \frac{1}{13} \\ 1 \end{bmatrix} (0, 13z) = q^{\frac{1}{104}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(13n^2 + n)} \end{array} \right. \quad (3.15)$$

be theta constants of order 13 and $\mathbf{A}(z) := (a_1(z), a_2(z), a_3(z), a_4(z), a_5(z), a_6(z))^T$. The significance of our six dimensional representation of $\text{PSL}(2, 13)$ comes from the following:

Proposition 3.2 (see [Y2], Proposition 2.5). *If $z \in \mathbb{H}$, then the following relations hold:*

$$\mathbf{A}(z+1) = e^{-\frac{3\pi i}{4}} T \mathbf{A}(z), \quad \mathbf{A}\left(-\frac{1}{z}\right) = e^{\frac{\pi i}{4}} \sqrt{z} S \mathbf{A}(z), \quad (3.16)$$

where S and T are given in (3.1) and (3.2), and $0 < \arg \sqrt{z} \leq \pi/2$.

Recall that the principal congruence subgroup of level 13 is the normal subgroup $\Gamma(13)$ of $\Gamma = \text{PSL}(2, \mathbb{Z})$ defined by the exact sequence $1 \rightarrow \Gamma(13) \rightarrow \Gamma(1) \xrightarrow{f} G \rightarrow 1$ where $f(\gamma) \equiv \gamma \pmod{13}$ for $\gamma \in \Gamma = \Gamma(1)$. There is a representation $\rho : \Gamma \rightarrow \text{PGL}(6, \mathbb{C})$

with kernel $\Gamma(13)$ defined as follows: if $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\rho(t) = T$ and $\rho(s) = S$. To see that such a representation exists, note that Γ is defined by the presentation $\langle s, t; s^2 = (st)^3 = 1 \rangle$ satisfied by s and t and we have proved that S and T satisfy these relations. Moreover, we have proved that G is defined by the presentation $\langle S, T; S^2 = T^{13} = (ST)^3 = 1 \rangle$. Let $p = st^{-1}s$ and $q = st^3$. Then

$$h := q^5 p^2 \cdot p^2 q^6 p^8 \cdot q^5 p^2 \cdot p^3 q = \begin{pmatrix} 4,428,249 & -10,547,030 \\ -11,594,791 & 27,616,019 \end{pmatrix}$$

satisfies that $\rho(h) = H$.

Let $x_i(z) = \eta(z)a_i(z)$, $y_i(z) = \eta^3(z)a_i(z)$ ($1 \leq i \leq 6$), $X(z) = (x_1(z), \dots, x_6(z))^T$ and $Y(z) = (y_1(z), \dots, y_6(z))^T$. Then $X(z) = \eta(z)\mathbf{A}(z)$ and $Y(z) = \eta^3(z)\mathbf{A}(z)$. Recall that $\eta(z)$ satisfies the following transformation formulas $\eta(z+1) = e^{\frac{\pi i}{12}}\eta(z)$ and $\eta(-\frac{1}{z}) = e^{-\frac{\pi i}{4}}\sqrt{z}\eta(z)$. By Proposition 3.2, we have $X(z+1) = e^{-\frac{2\pi i}{3}}\rho(t)X(z)$, $X(-\frac{1}{z}) = z\rho(s)X(z)$, $Y(z+1) = e^{-\frac{\pi i}{2}}\rho(t)Y(z)$ and $Y(-\frac{1}{z}) = e^{-\frac{\pi i}{2}}z^2\rho(s)Y(z)$.

Define $j(\gamma, z) := cz + d$ if $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Hence, $X(\gamma(z)) = u(\gamma)j(\gamma, z)\rho(\gamma)X(z)$ and $Y(\gamma(z)) = v(\gamma)j(\gamma, z)^2\rho(\gamma)Y(z)$ for $\gamma \in \Gamma(1)$, where $u(\gamma) = 1, \omega$ or ω^2 with $\omega = e^{\frac{2\pi i}{3}}$ and $v(\gamma) = \pm 1$ or $\pm i$. Since $\Gamma(13) = \ker \rho$, we have $X(\gamma(z)) = u(\gamma)j(\gamma, z)X(z)$ and $Y(\gamma(z)) = v(\gamma)j(\gamma, z)^2Y(z)$ for $\gamma \in \Gamma(13)$. This means that the functions $x_1(z), \dots, x_6(z)$ are modular forms of weight one for $\Gamma(13)$ with the same multiplier $u(\gamma) = 1, \omega$ or ω^2 and $y_1(z), \dots, y_6(z)$ are modular forms of weight two for $\Gamma(13)$ with the same multiplier $v(\gamma) = \pm 1$ or $\pm i$.

From now on, we will use the following abbreviation $\mathbb{A}_j = \mathbb{A}_j(a_1(z), \dots, a_6(z))$ ($0 \leq j \leq 6$), $\mathbb{D}_j = \mathbb{D}_j(a_1(z), \dots, a_6(z))$ ($j = 0, 1, \dots, 12, \infty$) and $\mathbb{G}_j = \mathbb{G}_j(a_1(z), \dots, a_6(z))$ ($0 \leq j \leq 12$). We have

$$\begin{aligned} \mathbb{A}_0 &= q^{\frac{1}{4}}(1 + O(q)), & \mathbb{A}_1 &= q^{\frac{17}{52}}(2 + O(q)), & \mathbb{A}_2 &= q^{\frac{29}{52}}(2 + O(q)), & \mathbb{A}_3 &= q^{\frac{49}{52}}(1 + O(q)), \\ \mathbb{A}_4 &= q^{\frac{25}{52}}(-1 + O(q)), & \mathbb{A}_5 &= q^{\frac{9}{52}}(-1 + O(q)), & \mathbb{A}_6 &= q^{\frac{1}{52}}(-1 + O(q)), \end{aligned}$$

and

$$\left\{ \begin{array}{l} \mathbb{D}_0 = q^{\frac{15}{8}}(1 + O(q)), \\ \mathbb{D}_\infty = q^{\frac{7}{8}}(-1 + O(q)), \\ \mathbb{D}_1 = q^{\frac{99}{104}}(2 + O(q)), \\ \mathbb{D}_2 = q^{\frac{3}{104}}(-1 + O(q)), \\ \mathbb{D}_3 = q^{\frac{11}{104}}(1 + O(q)), \\ \mathbb{D}_4 = q^{\frac{19}{104}}(-2 + O(q)), \\ \mathbb{D}_5 = q^{\frac{27}{104}}(-1 + O(q)), \end{array} \right. \left\{ \begin{array}{l} \mathbb{D}_6 = q^{\frac{35}{104}}(-1 + O(q)), \\ \mathbb{D}_7 = q^{\frac{43}{104}}(1 + O(q)), \\ \mathbb{D}_8 = q^{\frac{51}{104}}(3 + O(q)), \\ \mathbb{D}_9 = q^{\frac{59}{104}}(-2 + O(q)), \\ \mathbb{D}_{10} = q^{\frac{67}{104}}(1 + O(q)), \\ \mathbb{D}_{11} = q^{\frac{75}{104}}(-4 + O(q)), \\ \mathbb{D}_{12} = q^{\frac{83}{104}}(-1 + O(q)). \end{array} \right.$$

Hence,

$$\left\{ \begin{array}{l} \mathbb{G}_0 = q^{\frac{7}{4}}(1 + O(q)), \\ \mathbb{G}_1 = q^{\frac{43}{52}}(13 + O(q)), \\ \mathbb{G}_2 = q^{\frac{47}{52}}(-22 + O(q)), \\ \mathbb{G}_3 = q^{\frac{51}{52}}(-21 + O(q)), \\ \mathbb{G}_4 = q^{\frac{3}{52}}(-1 + O(q)), \\ \mathbb{G}_5 = q^{\frac{7}{52}}(2 + O(q)), \\ \mathbb{G}_6 = q^{\frac{11}{52}}(2 + O(q)), \end{array} \right. \left\{ \begin{array}{l} \mathbb{G}_7 = q^{\frac{15}{52}}(-2 + O(q)), \\ \mathbb{G}_8 = q^{\frac{19}{52}}(-8 + O(q)), \\ \mathbb{G}_9 = q^{\frac{23}{52}}(6 + O(q)), \\ \mathbb{G}_{10} = q^{\frac{27}{52}}(1 + O(q)), \\ \mathbb{G}_{11} = q^{\frac{31}{52}}(-8 + O(q)), \\ \mathbb{G}_{12} = q^{\frac{35}{52}}(17 + O(q)). \end{array} \right.$$

Note that

$$\begin{aligned} w_\nu &= (\mathbb{A}_0 + \zeta^\nu \mathbb{A}_1 + \zeta^{4\nu} \mathbb{A}_2 + \zeta^{9\nu} \mathbb{A}_3 + \zeta^{3\nu} \mathbb{A}_4 + \zeta^{12\nu} \mathbb{A}_5 + \zeta^{10\nu} \mathbb{A}_6)^2 \\ &= \mathbb{A}_0^2 + 2(\mathbb{A}_1 \mathbb{A}_5 + \mathbb{A}_2 \mathbb{A}_3 + \mathbb{A}_4 \mathbb{A}_6) + \\ &\quad + 2\zeta^\nu (\mathbb{A}_0 \mathbb{A}_1 + \mathbb{A}_2 \mathbb{A}_6) + 2\zeta^{3\nu} (\mathbb{A}_0 \mathbb{A}_4 + \mathbb{A}_2 \mathbb{A}_5) + 2\zeta^{9\nu} (\mathbb{A}_0 \mathbb{A}_3 + \mathbb{A}_5 \mathbb{A}_6) + \\ &\quad + 2\zeta^{12\nu} (\mathbb{A}_0 \mathbb{A}_5 + \mathbb{A}_3 \mathbb{A}_4) + 2\zeta^{10\nu} (\mathbb{A}_0 \mathbb{A}_6 + \mathbb{A}_1 \mathbb{A}_3) + 2\zeta^{4\nu} (\mathbb{A}_0 \mathbb{A}_2 + \mathbb{A}_1 \mathbb{A}_4) + \\ &\quad + \zeta^{2\nu} (\mathbb{A}_1^2 + 2\mathbb{A}_4 \mathbb{A}_5) + \zeta^{5\nu} (\mathbb{A}_3^2 + 2\mathbb{A}_1 \mathbb{A}_2) + \zeta^{6\nu} (\mathbb{A}_4^2 + 2\mathbb{A}_3 \mathbb{A}_6) + \\ &\quad + \zeta^{11\nu} (\mathbb{A}_5^2 + 2\mathbb{A}_1 \mathbb{A}_6) + \zeta^{8\nu} (\mathbb{A}_2^2 + 2\mathbb{A}_3 \mathbb{A}_5) + \zeta^{7\nu} (\mathbb{A}_6^2 + 2\mathbb{A}_4 \mathbb{A}_2), \end{aligned}$$

where

$$\left\{ \begin{array}{l} \mathbb{A}_0^2 + 2(\mathbb{A}_1 \mathbb{A}_5 + \mathbb{A}_2 \mathbb{A}_3 + \mathbb{A}_4 \mathbb{A}_6) = q^{\frac{1}{2}}(-1 + O(q)), \\ \mathbb{A}_0 \mathbb{A}_1 + \mathbb{A}_2 \mathbb{A}_6 = q^{\frac{41}{26}}(-3 + O(q)), \\ \mathbb{A}_0 \mathbb{A}_4 + \mathbb{A}_2 \mathbb{A}_5 = q^{\frac{19}{26}}(-3 + O(q)), \\ \mathbb{A}_0 \mathbb{A}_3 + \mathbb{A}_5 \mathbb{A}_6 = q^{\frac{5}{26}}(1 + O(q)), \\ \mathbb{A}_0 \mathbb{A}_5 + \mathbb{A}_3 \mathbb{A}_4 = q^{\frac{11}{26}}(-1 + O(q)), \\ \mathbb{A}_0 \mathbb{A}_6 + \mathbb{A}_1 \mathbb{A}_3 = q^{\frac{7}{26}}(-1 + O(q)), \\ \mathbb{A}_0 \mathbb{A}_2 + \mathbb{A}_1 \mathbb{A}_4 = q^{\frac{47}{26}}(-1 + O(q)), \\ \mathbb{A}_1^2 + 2\mathbb{A}_4 \mathbb{A}_5 = q^{\frac{17}{26}}(6 + O(q)), \\ \mathbb{A}_3^2 + 2\mathbb{A}_1 \mathbb{A}_2 = q^{\frac{23}{26}}(8 + O(q)), \\ \mathbb{A}_4^2 + 2\mathbb{A}_3 \mathbb{A}_6 = q^{\frac{25}{26}}(-1 + O(q)), \\ \mathbb{A}_5^2 + 2\mathbb{A}_1 \mathbb{A}_6 = q^{\frac{9}{26}}(-3 + O(q)), \\ \mathbb{A}_2^2 + 2\mathbb{A}_3 \mathbb{A}_5 = q^{\frac{29}{26}}(2 + O(q)), \\ \mathbb{A}_6^2 + 2\mathbb{A}_4 \mathbb{A}_2 = q^{\frac{1}{26}}(1 + O(q)). \end{array} \right.$$

Proof of Theorem 1.1. Let

$$\Phi_{20} = w_0^5 + w_1^5 + \cdots + w_{12}^5 + w_\infty^5.$$

As a polynomial in six variables, $\Phi_{20}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Moreover, for $\gamma \in \Gamma(1)$,

$$\begin{aligned} \Phi_{20}(Y(\gamma(z))^T) &= \Phi_{20}(v(\gamma)j(\gamma, z)^2(\rho(\gamma)Y(z))^T) \\ &= v(\gamma)^{20}j(\gamma, z)^{40}\Phi_{20}((\rho(\gamma)Y(z))^T) = j(\gamma, z)^{40}\Phi_{20}((\rho(\gamma)Y(z))^T). \end{aligned}$$

Note that $\rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G$ and Φ_{20} is a G -invariant polynomial, we have

$$\Phi_{20}(Y(\gamma(z))^T) = j(\gamma, z)^{40}\Phi_{20}(Y(z)^T), \quad \text{for } \gamma \in \Gamma(1).$$

This implies that $\Phi_{20}(y_1(z), \dots, y_6(z))$ is a modular form of weight 40 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{20}(a_1(z), \dots, a_6(z)) &= 13^5 q^{\frac{5}{2}} (1 + O(q))^5 + \\ &+ \sum_{\nu=0}^{12} [q^{\frac{1}{2}} (-1 + O(q)) + \\ &+ 2\zeta^\nu q^{\frac{41}{26}} (-3 + O(q)) + 2\zeta^{3\nu} q^{\frac{19}{26}} (-3 + O(q)) + 2\zeta^{9\nu} q^{\frac{5}{26}} (1 + O(q)) + \\ &+ 2\zeta^{12\nu} q^{\frac{11}{26}} (-1 + O(q)) + 2\zeta^{10\nu} q^{\frac{7}{26}} (-1 + O(q)) + 2\zeta^{4\nu} q^{\frac{47}{26}} (-1 + O(q)) + \\ &+ \zeta^{2\nu} q^{\frac{17}{26}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{23}{26}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{25}{26}} (-1 + O(q)) + \\ &+ \zeta^{11\nu} q^{\frac{9}{26}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{29}{26}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{1}{26}} (1 + O(q))]^5. \end{aligned}$$

We will calculate the $q^{\frac{1}{2}}$ -term which is the lowest degree. For the partition $13 = 4 \cdot 1 + 9$, the corresponding term is

$$\binom{5}{4, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^4 \cdot (-3) \zeta^{11\nu} q^{\frac{9}{26}} = -15q^{\frac{1}{2}}.$$

For the partition $13 = 3 \cdot 1 + 2 \cdot 5$, the corresponding term is

$$\binom{5}{3, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^3 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^2 = 40q^{\frac{1}{2}}.$$

Hence, for $\Phi_{20}(y_1(z), \dots, y_6(z))$ which is a modular form for $\Gamma(1)$ with weight 40, the lowest degree term is given by

$$(-15 + 40)q^{\frac{1}{2}} \cdot q^{\frac{3}{24} \cdot 20} = 25q^3.$$

Thus,

$$\Phi_{20}(y_1(z), \dots, y_6(z)) = q^3(13 \cdot 25 + O(q)).$$

The leading term of $\Phi_{20}(y_1(z), \dots, y_6(z))$ together with its weight 40 suffice to identify this modular form with $\Phi_{20}(y_1(z), \dots, y_6(z)) = 13 \cdot 25 \Delta(z)^3 E_4(z)$. Consequently,

$$\Phi_{20}(x_1(z), \dots, x_6(z)) = 13 \cdot 25 \Delta(z)^3 E_4(z) / \eta(z)^{40} = 13 \cdot 25 \eta(z)^8 \Delta(z) E_4(z).$$

Let

$$\Phi_{18} = \delta_0^3 + \delta_1^3 + \dots + \delta_{12}^3 + \delta_\infty^3.$$

As a polynomial in six variables, $\Phi_{18}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Moreover, for $\gamma \in \Gamma(1)$,

$$\begin{aligned} \Phi_{18}(X(\gamma(z))^T) &= \Phi_{18}(u(\gamma)j(\gamma, z)(\rho(\gamma)X(z))^T) \\ &= u(\gamma)^{18} j(\gamma, z)^{18} \Phi_{18}((\rho(\gamma)X(z))^T) = j(\gamma, z)^{18} \Phi_{18}((\rho(\gamma)X(z))^T). \end{aligned}$$

Note that $\rho(\gamma) \in \langle \rho(s), \rho(t) \rangle = G$ and Φ_{18} is a G -invariant polynomial, we have

$$\Phi_{18}(X(\gamma(z))^T) = j(\gamma, z)^{18} \Phi_{18}(X(z)^T), \quad \text{for } \gamma \in \Gamma(1).$$

This implies that $\Phi_{18}(x_1(z), \dots, x_6(z))$ is a modular form of weight 18 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{18}(a_1(z), \dots, a_6(z)) &= 13^6 q^{\frac{21}{4}} (1 + O(q))^3 + \\ &+ \sum_{\nu=0}^{12} [-13 q^{\frac{7}{4}} (1 + O(q)) + \\ &+ \zeta^\nu q^{\frac{43}{52}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{47}{52}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{51}{52}} (-21 + O(q)) + \\ &+ \zeta^{4\nu} q^{\frac{3}{52}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{7}{52}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{11}{52}} (2 + O(q)) + \\ &+ \zeta^{7\nu} q^{\frac{15}{52}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{19}{52}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{23}{52}} (6 + O(q)) + \\ &+ \zeta^{10\nu} q^{\frac{27}{52}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{31}{52}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{35}{52}} (17 + O(q))]^3. \end{aligned}$$

We will calculate the $q^{\frac{1}{4}}$ -term which is the lowest degree. For the partition $13 = 2 \cdot 3 + 7$, the corresponding term is

$$\binom{3}{2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 = 6q^{\frac{1}{4}}.$$

Hence, for $\Phi_{18}(x_1(z), \dots, x_6(z))$ which is a modular form for $\Gamma(1)$ with weight 18, the lowest degree term is given by $6q^{\frac{1}{4}} \cdot q^{\frac{18}{4}} = 6q$. Thus,

$$\Phi_{18}(x_1(z), \dots, x_6(z)) = q(13 \cdot 6 + O(q)).$$

The leading term of $\Phi_{18}(x_1(z), \dots, x_6(z))$ together with its weight 18 suffice to identify this modular form with

$$\Phi_{18}(x_1(z), \dots, x_6(z)) = 13 \cdot 6\Delta(z)E_6(z).$$

Let

$$\Phi_{12} = \delta_0^2 + \delta_1^2 + \dots + \delta_{12}^2 + \delta_\infty^2.$$

As a polynomial in six variables, $\Phi_{12}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Similarly as above, we can show that $\Phi_{12}(x_1(z), \dots, x_6(z))$ is a modular form of weight 12 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{12}(a_1(z), \dots, a_6(z)) &= 13^4 q^{\frac{7}{2}} (1 + O(q))^2 + \\ &+ \sum_{\nu=0}^{12} [-13q^{\frac{7}{2}} (1 + O(q)) + \\ &+ \zeta^\nu q^{\frac{43}{52}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{47}{52}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{51}{52}} (-21 + O(q)) + \\ &+ \zeta^{4\nu} q^{\frac{3}{52}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{7}{52}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{11}{52}} (2 + O(q)) + \\ &+ \zeta^{7\nu} q^{\frac{15}{52}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{19}{52}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{23}{52}} (6 + O(q)) + \\ &+ \zeta^{10\nu} q^{\frac{27}{52}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{31}{52}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{35}{52}} (17 + O(q))]^2. \end{aligned}$$

We will calculate the $q^{\frac{1}{2}}$ -term which is the lowest degree. For the partition $26 = 3 + 23$, the corresponding term is

$$\binom{2}{1, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = -12q^{\frac{1}{2}}.$$

For the partition $26 = 7 + 19$, the corresponding term is

$$\binom{2}{1, 1} \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = -32q^{\frac{1}{2}}.$$

For the partition $26 = 11 + 15$, the corresponding term is

$$\binom{2}{1, 1} \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) = -8q^{\frac{1}{2}}.$$

Hence, for $\Phi_{12}(x_1(z), \dots, x_6(z))$ which is a modular form for $\Gamma(1)$ with weight 12, the lowest degree term is given by $(-12 - 32 - 8)q^{\frac{1}{2}} \cdot q^{\frac{12}{24}} = -52q$. Thus,

$$\Phi_{12}(x_1(z), \dots, x_6(z)) = q(-13 \cdot 52 + O(q)).$$

The leading term of $\Phi_{12}(x_1(z), \dots, x_6(z))$ together with its weight 12 suffice to identify this modular form with

$$\Phi_{12}(x_1(z), \dots, x_6(z)) = -13 \cdot 52\Delta(z).$$

Let

$$\Phi_{30} = \delta_0^5 + \delta_1^5 + \dots + \delta_{12}^5 + \delta_\infty^5.$$

As a polynomial in six variables, $\Phi_{30}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Similarly as above, we can show that $\Phi_{30}(x_1(z), \dots, x_6(z))$ is a modular form of weight 30 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{30}(a_1(z), \dots, a_6(z)) &= 13^{10} q^{\frac{35}{4}} (1 + O(q))^5 + \\ &+ \sum_{\nu=0}^{12} [-13q^{\frac{7}{4}} (1 + O(q)) + \\ &+ \zeta^\nu q^{\frac{43}{52}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{47}{52}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{51}{52}} (-21 + O(q)) + \\ &+ \zeta^{4\nu} q^{\frac{3}{52}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{7}{52}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{11}{52}} (2 + O(q)) + \\ &+ \zeta^{7\nu} q^{\frac{15}{52}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{19}{52}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{23}{52}} (6 + O(q)) + \\ &+ \zeta^{10\nu} q^{\frac{27}{52}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{31}{52}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{35}{52}} (17 + O(q))]^5. \end{aligned}$$

We will calculate the $q^{\frac{3}{4}}$ -term which is the lowest degree. (1) For the partition $39 = 4 \cdot 3 + 27$, the corresponding term is

$$\binom{5}{4, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{10\nu} q^{\frac{27}{52}} = 5q^{\frac{3}{4}}.$$

(2) For the partition $39 = 3 \cdot 3 + 7 + 23$, the corresponding term is

$$\binom{5}{3, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = -240q^{\frac{3}{4}}.$$

(3) For the partition $39 = 3 \cdot 3 + 11 + 19$, the corresponding term is

$$\binom{5}{3, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = 320q^{\frac{3}{4}}.$$

(4) For the partition $39 = 3 \cdot 3 + 2 \cdot 15$, the corresponding term is

$$\binom{5}{3, 2} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^2 = -40q^{\frac{3}{4}}.$$

(5) For the partition $39 = 2 \cdot 3 + 3 \cdot 11$, the corresponding term is

$$\binom{5}{2, 3} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^3 = 80q^{\frac{3}{4}}.$$

(6) For the partition $39 = 2 \cdot 3 + 2 \cdot 7 + 19$, the corresponding term is

$$\binom{5}{2, 2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = -960q^{\frac{3}{4}}.$$

(7) For the partition $39 = 2 \cdot 3 + 7 + 11 + 15$, the corresponding term is

$$\binom{5}{2, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) = -480q^{\frac{3}{4}}.$$

(8) For the partition $39 = 1 \cdot 3 + 3 \cdot 7 + 15$, the corresponding term is

$$\binom{5}{1, 3, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) = 320q^{\frac{3}{4}}.$$

(9) For the partition $39 = 1 \cdot 3 + 2 \cdot 7 + 2 \cdot 11$, the corresponding term is

$$\binom{5}{1, 2, 2} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 = -480q^{\frac{3}{4}}.$$

(10) For the partition $39 = 4 \cdot 7 + 11$, the corresponding term is

$$\binom{5}{4, 1} (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^4 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 = 160q^{\frac{3}{4}}.$$

Hence, for $\Phi_{30}(x_1(z), \dots, x_6(z))$ which is a modular form for $\Gamma(1)$ with weight 30, the lowest degree term is given by

$$(5 - 240 + 320 - 40 + 80 - 960 - 480 + 320 - 480 + 160)q^{\frac{3}{4}} \cdot q^{\frac{30}{24}} = -1315q^2.$$

Thus,

$$\Phi_{30}(x_1(z), \dots, x_6(z)) = q^2(-13 \cdot 1315 + O(q)).$$

The leading term of $\Phi_{30}(x_1(z), \dots, x_6(z))$ together with its weight 30 suffice to identify this modular form with

$$\Phi_{30}(x_1(z), \dots, x_6(z)) = -13 \cdot 1315 \Delta(z)^2 E_6(z).$$

Up to a constant, we revise the definition of Φ_{12} , Φ_{18} , Φ_{20} and Φ_{30} :

$$\Phi_{12} = -\frac{1}{13 \cdot 52} \left(\sum_{\nu=0}^{12} \delta_{\nu}^2 + \delta_{\infty}^2 \right), \quad \Phi_{18} = \frac{1}{13 \cdot 6} \left(\sum_{\nu=0}^{12} \delta_{\nu}^3 + \delta_{\infty}^3 \right), \quad (3.17)$$

$$\Phi_{20} = \frac{1}{13 \cdot 25} \left(\sum_{\nu=0}^{12} w_{\nu}^5 + w_{\infty}^5 \right), \quad \Phi_{30} = -\frac{1}{13 \cdot 1315} \left(\sum_{\nu=0}^{12} \delta_{\nu}^5 + \delta_{\infty}^5 \right). \quad (3.18)$$

Consequently,

$$\begin{cases} \Phi_{12}(x_1(z), \dots, x_6(z)) = \Delta(z), \\ \Phi_{18}(x_1(z), \dots, x_6(z)) = \Delta(z)E_6(z), \\ \Phi_{20}(x_1(z), \dots, x_6(z)) = \eta(z)^8 \Delta(z)E_4(z), \\ \Phi_{30}(x_1(z), \dots, x_6(z)) = \Delta(z)^2 E_6(z). \end{cases} \quad (3.19)$$

From now on, we will use the following abbreviation $\Phi_j = \Phi_j(x_1(z), \dots, x_6(z))$ for $j = 12, 18, 20$ and 30 . The relations

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = \frac{\Phi_{20}^3}{\Phi_{12}^5}, \quad j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = \frac{\Phi_{30}^2}{\Phi_{12}^5} = \frac{\Phi_{18}^2}{\Phi_{12}^3} \quad (3.20)$$

give the equations

$$\Phi_{20}^3 - \Phi_{30}^2 = 1728\Phi_{12}^5, \quad \Phi_{20}^3 - \Phi_{12}^2\Phi_{18}^2 = 1728\Phi_{12}^5. \quad (3.21)$$

This completes the proof of Theorem 1.1. \square

Let us recall some facts about exotic spheres (see [Hi2]). A k -dimensional compact oriented differentiable manifold is called a k -sphere if it is homeomorphic to the k -dimensional standard sphere. A k -sphere not diffeomorphic to the standard k -sphere is said to be exotic. The first exotic sphere was discovered by Milnor in 1956 (see [Mi1] and [Mi4]). Two k -spheres are called equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of k -spheres constitute for $k \geq 5$ a finite abelian group Θ_k under the connected sum operation. Θ_k contains the subgroup bP_{k+1} of those k -spheres which bound a parallelizable manifold. bP_{4m} ($m \geq 2$) is cyclic of order $2^{2m-2}(2^{2m-1}-1)$ numerator $(4B_m/m)$, where B_m is the m -th Bernoulli number. Let g_m be the Milnor generator of bP_{4m} . If a $(4m-1)$ -sphere Σ bounds a parallelizable manifold B of dimension $4m$, then the signature $\tau(B)$ of the intersection form of B is divisible by 8 and $\Sigma = \frac{\tau(B)}{8}g_m$. For $m = 2$ we have $bP_8 = \Theta_7 = \mathbb{Z}/28\mathbb{Z}$. All these results are due to Milnor-Kervaire (see [KM]). In particular,

$$\sum_{i=0}^{2m} z_i \bar{z}_i = 1, \quad z_0^3 + z_1^{6k-1} + z_2^2 + \dots + z_{2m}^2 = 0$$

is a $(4m-1)$ -sphere embedded in $S^{4m+1} \subset \mathbb{C}^{2n+1}$ which represents the element $(-1)^m k \cdot g_m \in bP_{4m}$. For $m = 2$ and $k = 1, 2, \dots, 28$ we get the 28 classes of 7-spheres. Theorem 1.1 shows that the higher dimensional liftings of two distinct symmetry groups and modular interpretations on the equation of E_8 -singularity give the same Milnor's standard generator of Θ_7 .

In fact, G is the symmetry group of regular maps $\{7, 3\}_{13}$, $\{13, 3\}_7$ and $\{13, 7\}_3$, which are the generalizations of regular polyhedra in \mathbb{R}^3 (see [CoM], p.140).

Maps	Vertices	Edges	Faces	Characteristic	Group
$\{7, 3\}_{13}$	182	273	78	-13	$\text{PSL}(2, 13)$
$\{13, 3\}_7$	182	273	42	-49	$\text{PSL}(2, 13)$
$\{13, 7\}_3$	78	273	42	-153	$\text{PSL}(2, 13)$

4. Modular parametrizations for some exceptional singularities and Fermat-Catalan conjecture

Proof of Theorem 1.2 and Corollary 1.3. Let

$$\Phi_{32} = w_0^8 + w_1^8 + \dots + w_{12}^8 + w_\infty^8.$$

As a polynomial in six variables, $\Phi_{32}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Similarly as above, we can show that $\Phi_{32}(y_1(z), \dots, y_6(z))$ is a modular form of weight 64 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{32}(a_1(z), \dots, a_6(z)) &= 13^8 q^4 (1 + O(q))^8 + \\ &+ \sum_{\nu=0}^{12} [q^{\frac{1}{2}} (-1 + O(q)) + \\ &+ 2\zeta^\nu q^{\frac{41}{26}} (-3 + O(q)) + 2\zeta^{3\nu} q^{\frac{19}{26}} (-3 + O(q)) + 2\zeta^{9\nu} q^{\frac{5}{26}} (1 + O(q)) + \\ &+ 2\zeta^{12\nu} q^{\frac{11}{26}} (-1 + O(q)) + 2\zeta^{10\nu} q^{\frac{7}{26}} (-1 + O(q)) + 2\zeta^{4\nu} q^{\frac{47}{26}} (-1 + O(q)) + \\ &+ \zeta^{2\nu} q^{\frac{17}{26}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{23}{26}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{25}{26}} (-1 + O(q)) + \\ &+ \zeta^{11\nu} q^{\frac{9}{26}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{29}{26}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{1}{26}} (1 + O(q))]^8. \end{aligned}$$

We will calculate the q -term which is the lowest degree. (1) For the partition $26 = 7 \cdot 1 + 19$, the corresponding term is

$$\binom{8}{7, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot 2\zeta^{3\nu} q^{\frac{19}{26}} \cdot (-3) = -48q.$$

(2) For the partition $26 = 6 \cdot 1 + 9 + 11$, the corresponding term is

$$\binom{8}{6, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot (\zeta^{11\nu} q^{\frac{9}{26}}) \cdot (-3) \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} (-1) = 336q.$$

(3) For the partition $26 = 6 \cdot 1 + 7 + 13$, the corresponding term is

$$\binom{8}{6, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1) \cdot q^{\frac{1}{2}} (-1) = 112q.$$

(4) For the partition $26 = 5 \cdot 1 + 5 + 7 + 9$, the corresponding term is

$$\binom{8}{5, 1, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^5 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} (-1) \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) = 4032q.$$

(5) For the partition $26 = 5 \cdot 1 + 2 \cdot 5 + 11$, the corresponding term is

$$\binom{8}{5, 2, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^5 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^2 \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} (-1) = -1344q.$$

(6) For the partition $26 = 5 \cdot 1 + 3 \cdot 7$, the corresponding term is

$$\binom{8}{5, 3} (\zeta^{7\nu} q^{\frac{1}{26}})^5 \cdot (2\zeta^{10\nu} q^{\frac{7}{26}})^3 \cdot (-1)^3 = -448q.$$

(7) For the partition $26 = 4 \cdot 1 + 3 \cdot 5 + 7$, the corresponding term is

$$\binom{8}{4, 3, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^4 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^3 \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} (-1) = -4480q.$$

Hence, for $\Phi_{32}(y_1(z), \dots, y_6(z))$ which is a modular form for $\Gamma(1)$ with weight 64, the lowest degree term is given by

$$(-48 + 336 + 112 + 4032 - 1344 - 448 - 4480)q \cdot q^{\frac{3}{24} \cdot 32} = -1840q^5.$$

Thus,

$$\Phi_{32}(y_1(z), \dots, y_6(z)) = q^5(-13 \cdot 1840 + O(q)).$$

The leading term of $\Phi_{32}(y_1(z), \dots, y_6(z))$ together with its weight 64 suffice to identify this modular form with $\Phi_{32}(y_1(z), \dots, y_6(z)) = -13 \cdot 1840 \Delta(z)^5 E_4(z)$. Consequently,

$$\Phi_{32}(x_1(z), \dots, x_6(z)) = -13 \cdot 1840 \Delta(z)^5 E_4(z) / \eta(z)^{64} = -13 \cdot 1840 \eta(z)^8 \Delta(z)^2 E_4(z).$$

Let

$$\Phi_{44} = w_0^{11} + w_1^{11} + \cdots + w_{12}^{11} + w_{\infty}^{11}.$$

As a polynomial in six variables, $\Phi_{44}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Similarly as above, we can show that $\Phi_{44}(y_1(z), \dots, y_6(z))$ is a modular form of weight 88 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{44}(a_1(z), \dots, a_6(z)) &= 13^{11} q^{\frac{11}{2}} (1 + O(q))^{11} + \\ &+ \sum_{\nu=0}^{12} [q^{\frac{1}{2}} (-1 + O(q)) + \\ &+ 2\zeta^{\nu} q^{\frac{41}{26}} (-3 + O(q)) + 2\zeta^{3\nu} q^{\frac{19}{26}} (-3 + O(q)) + 2\zeta^{9\nu} q^{\frac{5}{26}} (1 + O(q)) + \\ &+ 2\zeta^{12\nu} q^{\frac{11}{26}} (-1 + O(q)) + 2\zeta^{10\nu} q^{\frac{7}{26}} (-1 + O(q)) + 2\zeta^{4\nu} q^{\frac{47}{26}} (-1 + O(q)) + \\ &+ \zeta^{2\nu} q^{\frac{17}{26}} (6 + O(q)) + \zeta^{5\nu} q^{\frac{23}{26}} (8 + O(q)) + \zeta^{6\nu} q^{\frac{25}{26}} (-1 + O(q)) + \\ &+ \zeta^{11\nu} q^{\frac{9}{26}} (-3 + O(q)) + \zeta^{8\nu} q^{\frac{29}{26}} (2 + O(q)) + \zeta^{7\nu} q^{\frac{1}{26}} (1 + O(q))]^{11}. \end{aligned}$$

We will calculate the $q^{\frac{3}{2}}$ -term which is the lowest degree. (1) For the partition $39 = 10 \cdot 1 + 29$, the corresponding term is

$$\binom{11}{10, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^{10} \cdot \zeta^{8\nu} q^{\frac{29}{26}} \cdot 2 = 22q^{\frac{3}{2}}.$$

(2) For the partition $39 = 9 \cdot 1 + 13 + 17$, the corresponding term is

$$\binom{11}{9, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^9 \cdot q^{\frac{1}{2}} \cdot (-1) \cdot \zeta^{2\nu} q^{\frac{17}{26}} \cdot 6 = -660q^{\frac{3}{2}}.$$

(3) For the partition $39 = 9 \cdot 1 + 11 + 19$, the corresponding term is

$$\binom{11}{9, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^9 \cdot 2\zeta^{3\nu} q^{\frac{19}{26}} \cdot (-3) \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1) = 1320q^{\frac{3}{2}}.$$

(4) For the partition $39 = 9 \cdot 1 + 7 + 23$, the corresponding term is

$$\binom{11}{9, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^9 \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1) \cdot \zeta^{5\nu} q^{\frac{23}{26}} \cdot 8 = -1760q^{\frac{3}{2}}.$$

(5) For the partition $39 = 9 \cdot 1 + 5 + 25$, the corresponding term is

$$\binom{11}{9, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^9 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot \zeta^{6\nu} q^{\frac{25}{26}} \cdot (-1) = -220q^{\frac{3}{2}}.$$

(6) For the partition $39 = 8 \cdot 1 + 13 + 2 \cdot 9$, the corresponding term is

$$\binom{11}{8, 1, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot q^{\frac{1}{2}} \cdot (-1) \cdot (\zeta^{11\nu} q^{\frac{9}{26}})^2 \cdot (-3)^2 = -4455q^{\frac{3}{2}}.$$

(7) For the partition $39 = 8 \cdot 1 + 13 + 7 + 11$, the corresponding term is

$$\binom{11}{8, 1, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot q^{\frac{1}{2}} \cdot (-1) \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1) \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1) = -3960q^{\frac{3}{2}}.$$

(8) For the partition $39 = 8 \cdot 1 + 2 \cdot 13 + 5$, the corresponding term is

$$\binom{11}{8, 2, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot (q^{\frac{1}{2}} \cdot (-1))^2 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} = 990q^{\frac{3}{2}}.$$

(9) For the partition $39 = 8 \cdot 1 + 19 + 5 + 7$, the corresponding term is

$$\binom{11}{8, 1, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot 2\zeta^{3\nu} q^{\frac{19}{26}} \cdot (-3) \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1) = 23760q^{\frac{3}{2}}.$$

(10) For the partition $39 = 8 \cdot 1 + 2 \cdot 11 + 9$, the corresponding term is

$$\binom{11}{8, 2, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot (2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1))^2 \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) = -5940q^{\frac{3}{2}}.$$

(11) For the partition $39 = 8 \cdot 1 + 5 + 17 + 9$, the corresponding term is

$$\binom{11}{8, 1, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot \zeta^{2\nu} q^{\frac{17}{26}} \cdot 6 \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) = -35640q^{\frac{3}{2}}.$$

(12) For the partition $39 = 8 \cdot 1 + 2 \cdot 7 + 17$, the corresponding term is

$$\binom{11}{8, 2, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^8 \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^2 \cdot \zeta^{2\nu} q^{\frac{17}{26}} \cdot 6 = 11880q^{\frac{3}{2}}.$$

(13) For the partition $39 = 7 \cdot 1 + 13 + 5 + 2 \cdot 7$, the corresponding term is

$$\binom{11}{7, 1, 1, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot q^{\frac{1}{2}} \cdot (-1) \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^2 = -31680q^{\frac{3}{2}}.$$

(14) For the partition $39 = 7 \cdot 1 + 13 + 2 \cdot 5 + 9$, the corresponding term is

$$\binom{11}{7, 1, 2, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot q^{\frac{1}{2}} \cdot (-1) \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^2 \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) = 47520q^{\frac{3}{2}}.$$

(15) For the partition $39 = 7 \cdot 1 + 5 + 11 + 7 + 9$, the corresponding term is

$$\begin{aligned} & \binom{11}{7, 1, 1, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1) \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1) \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) \\ &= -190080q^{\frac{3}{2}}. \end{aligned}$$

(16) For the partition $39 = 7 \cdot 1 + 17 + 3 \cdot 5$, the corresponding term is

$$\binom{11}{7, 3, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot \zeta^{2\nu} q^{\frac{17}{26}} \cdot 6 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^3 = 63360q^{\frac{3}{2}}.$$

(17) For the partition $39 = 7 \cdot 1 + 5 + 3 \cdot 9$, the corresponding term is

$$\binom{11}{7, 3, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot (\zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3))^3 = -71280q^{\frac{3}{2}}.$$

(18) For the partition $39 = 7 \cdot 1 + 2 \cdot 5 + 2 \cdot 11$, the corresponding term is

$$\binom{11}{7, 2, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^2 \cdot (2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1))^2 = 31680q^{\frac{3}{2}}.$$

(19) For the partition $39 = 7 \cdot 1 + 11 + 3 \cdot 7$, the corresponding term is

$$\binom{11}{7, 3, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1) \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^3 = 21120q^{\frac{3}{2}}.$$

(20) For the partition $39 = 7 \cdot 1 + 2 \cdot 9 + 2 \cdot 7$, the corresponding term is

$$\binom{11}{7, 2, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^7 \cdot (\zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3))^2 \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^2 = 71280q^{\frac{3}{2}}.$$

(21) For the partition $39 = 6 \cdot 1 + 13 + 4 \cdot 5$, the corresponding term is

$$\binom{11}{6, 1, 4} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot q^{\frac{1}{2}} \cdot (-1) \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^4 = -36960q^{\frac{3}{2}}.$$

(22) For the partition $39 = 6 \cdot 1 + 5 + 4 \cdot 7$, the corresponding term is

$$\binom{11}{6, 1, 4} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot 2\zeta^{9\nu} q^{\frac{5}{26}} \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^4 = 73920q^{\frac{3}{2}}.$$

(23) For the partition $39 = 6 \cdot 1 + 2 \cdot 5 + 2 \cdot 7 + 9$, the corresponding term is

$$\binom{11}{6, 2, 2, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^2 \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^2 \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) = -665280q^{\frac{3}{2}}.$$

(24) For the partition $39 = 6 \cdot 1 + 3 \cdot 5 + 2 \cdot 9$, the corresponding term is

$$\binom{11}{6, 3, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^3 \cdot (\zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3))^2 = 332640q^{\frac{3}{2}}.$$

(25) For the partition $39 = 6 \cdot 1 + 3 \cdot 5 + 7 + 11$, the corresponding term is

$$\binom{11}{6, 3, 1, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^6 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^3 \cdot 2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1) \cdot 2\zeta^{12\nu} q^{\frac{11}{26}} \cdot (-1) = 295680q^{\frac{3}{2}}.$$

(26) For the partition $39 = 5 \cdot 1 + 5 \cdot 5 + 9$, the corresponding term is

$$\binom{11}{5, 5, 1} (\zeta^{7\nu} q^{\frac{1}{26}})^5 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^5 \cdot \zeta^{11\nu} q^{\frac{9}{26}} \cdot (-3) = -266112q^{\frac{3}{2}}.$$

(27) For the partition $39 = 5 \cdot 1 + 4 \cdot 5 + 2 \cdot 7$, the corresponding term is

$$\binom{11}{5, 4, 2} (\zeta^{7\nu} q^{\frac{1}{26}})^5 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^4 \cdot (2\zeta^{10\nu} q^{\frac{7}{26}} \cdot (-1))^2 = 443520q^{\frac{3}{2}}.$$

(28) For the partition $39 = 4 \cdot 1 + 7 \cdot 5$, the corresponding term is

$$\binom{11}{4, 7} (\zeta^{7\nu} q^{\frac{1}{26}})^4 \cdot (2\zeta^{9\nu} q^{\frac{5}{26}})^7 = 42240q^{\frac{3}{2}}.$$

Hence, for $\Phi_{44}(y_1(z), \dots, y_6(z))$ which is a modular form for $\Gamma(1)$ with weight 88, the lowest degree term is given by

$$\begin{aligned} & (22 - 660 + 1320 - 1760 - 220 - 4455 - 3960 + 990 + 23760 - 5940 - 35640 + \\ & + 11880 - 31680 + 47520 - 190080 + 63360 - 71280 + 31680 + 21120 + 71280 + \\ & - 36960 + 73920 - 665280 + 332640 + 295680 - 266112 + 443520 + 42240)q^{\frac{3}{2}} \cdot q^{\frac{3}{24} \cdot 44} \\ & = 146905q^7. \end{aligned}$$

Thus,

$$\Phi_{44}(y_1(z), \dots, y_6(z)) = q^7(13 \cdot 146905 + O(q)).$$

The leading term of $\Phi_{44}(y_1(z), \dots, y_6(z))$ together with its weight 88 suffice to identify this modular form with $\Phi_{44}(y_1(z), \dots, y_6(z)) = 13 \cdot 146905\Delta(z)^7 E_4(z)$. Consequently,

$$\Phi_{44}(x_1(z), \dots, x_6(z)) = 13 \cdot 146905\Delta(z)^7 E_4(z)/\eta(z)^{88} = 13 \cdot 146905\eta(z)^8 \Delta(z)^3 E_4(z).$$

Let

$$\Phi_{42} = \delta_0^7 + \delta_1^7 + \dots + \delta_{12}^7 + \delta_\infty^7.$$

As a polynomial in six variables, $\Phi_{42}(z_1, z_2, z_3, z_4, z_5, z_6)$ is a G -invariant polynomial. Similarly as above, we can show that $\Phi_{42}(x_1(z), \dots, x_6(z))$ is a modular form of weight 42 for the full modular group $\Gamma(1)$. Moreover, we will show that it is a cusp form. In fact,

$$\begin{aligned} \Phi_{42}(a_1(z), \dots, a_6(z)) &= 13^{14} q^{\frac{49}{4}} (1 + O(q))^7 + \\ &+ \sum_{\nu=0}^{12} [-13q^{\frac{7}{4}} (1 + O(q)) + \\ &+ \zeta^\nu q^{\frac{43}{52}} (13 + O(q)) + \zeta^{2\nu} q^{\frac{47}{52}} (-22 + O(q)) + \zeta^{3\nu} q^{\frac{51}{52}} (-21 + O(q)) + \\ &+ \zeta^{4\nu} q^{\frac{3}{52}} (-1 + O(q)) + \zeta^{5\nu} q^{\frac{7}{52}} (2 + O(q)) + \zeta^{6\nu} q^{\frac{11}{52}} (2 + O(q)) + \\ &+ \zeta^{7\nu} q^{\frac{15}{52}} (-2 + O(q)) + \zeta^{8\nu} q^{\frac{19}{52}} (-8 + O(q)) + \zeta^{9\nu} q^{\frac{23}{52}} (6 + O(q)) + \\ &+ \zeta^{10\nu} q^{\frac{27}{52}} (1 + O(q)) + \zeta^{11\nu} q^{\frac{31}{52}} (-8 + O(q)) + \zeta^{12\nu} q^{\frac{35}{52}} (17 + O(q))]^7. \end{aligned}$$

Note that $3, 7, 11, 15, 19, 23, 27, 31, 35 \equiv 3 \pmod{4}$, and $7(4k+3) \equiv 1 \pmod{4}$, but $39 \equiv 3 \pmod{4}$. This implies that $q^{\frac{5}{4}}$ -term is the lowest degree. (1) For the partition $65 = 6 \cdot 3 + 47$, the corresponding term is

$$\binom{7}{6, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^6 \cdot \zeta^{2\nu} q^{\frac{47}{52}} \cdot (-22) = -154q^{\frac{5}{4}}.$$

(2) For the partition $65 = 5 \cdot 3 + 7 + 43$, the corresponding term is

$$\binom{7}{5, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^5 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^\nu q^{\frac{43}{52}} \cdot 13 = -1092q^{\frac{5}{4}}.$$

(3) For the partition $65 = 5 \cdot 3 + 15 + 35$, the corresponding term is

$$\binom{7}{5, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^5 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{12\nu} q^{\frac{35}{52}} \cdot 17 = 1428q^{\frac{5}{4}}.$$

(4) For the partition $65 = 5 \cdot 3 + 19 + 31$, the corresponding term is

$$\binom{7}{5, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^5 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) \cdot \zeta^{11\nu} q^{\frac{31}{52}} \cdot (-8) = -2688q^{\frac{5}{4}}.$$

(5) For the partition $65 = 5 \cdot 3 + 23 + 27$, the corresponding term is

$$\binom{7}{5, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^5 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 \cdot \zeta^{10\nu} q^{\frac{27}{52}} = -252q^{\frac{5}{4}}.$$

(6) For the partition $65 = 4 \cdot 3 + 7 + 11 + 35$, the corresponding term is

$$\binom{7}{4, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{12\nu} q^{\frac{35}{52}} \cdot 17 = 14280q^{\frac{5}{4}}.$$

(7) For the partition $65 = 4 \cdot 3 + 7 + 15 + 31$, the corresponding term is

$$\binom{7}{4, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{11\nu} q^{\frac{31}{52}} \cdot (-8) = 6720q^{\frac{5}{4}}.$$

(8) For the partition $65 = 4 \cdot 3 + 7 + 19 + 27$, the corresponding term is

$$\binom{7}{4, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) \cdot \zeta^{10\nu} q^{\frac{27}{52}} = -3360q^{\frac{5}{4}}.$$

(9) For the partition $65 = 4 \cdot 3 + 7 + 2 \cdot 23$, the corresponding term is

$$\binom{7}{4, 1, 2} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot (\zeta^{9\nu} q^{\frac{23}{52}} \cdot 6)^2 = 7560q^{\frac{5}{4}}.$$

(10) For the partition $65 = 4 \cdot 3 + 11 + 15 + 27$, the corresponding term is

$$\binom{7}{4, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{10\nu} q^{\frac{27}{52}} = -840q^{\frac{5}{4}}.$$

(11) For the partition $65 = 4 \cdot 3 + 11 + 19 + 23$, the corresponding term is

$$\binom{7}{4, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = -20160q^{\frac{5}{4}}.$$

(12) For the partition $65 = 4 \cdot 3 + 2 \cdot 11 + 31$, the corresponding term is

$$\binom{7}{4, 2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot \zeta^{11\nu} q^{\frac{31}{52}} \cdot (-8) = -3360q^{\frac{5}{4}}.$$

(13) For the partition $65 = 4 \cdot 3 + 15 + 2 \cdot 19$, the corresponding term is

$$\binom{7}{4, 1, 2} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot (\zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8))^2 = -13440q^{\frac{5}{4}}.$$

(14) For the partition $65 = 4 \cdot 3 + 2 \cdot 15 + 23$, the corresponding term is

$$\binom{7}{4, 2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^4 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^2 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = 2520q^{\frac{5}{4}}.$$

(15) For the partition $65 = 3 \cdot 3 + 7 + 11 + 2 \cdot 19$, the corresponding term is

$$\binom{7}{3, 1, 1, 2} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot (\zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8))^2 = -107520q^{\frac{5}{4}}.$$

(16) For the partition $65 = 3 \cdot 3 + 7 + 11 + 15 + 23$, the corresponding term is

$$\binom{7}{3, 1, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = 40320q^{\frac{5}{4}}.$$

(17) For the partition $65 = 3 \cdot 3 + 7 + 2 \cdot 11 + 27$, the corresponding term is

$$\binom{7}{3, 1, 2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot \zeta^{10\nu} q^{\frac{27}{52}} = -3360q^{\frac{5}{4}}.$$

(18) For the partition $65 = 3 \cdot 3 + 7 + 2 \cdot 15 + 19$, the corresponding term is

$$\binom{7}{3, 1, 2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = 26880q^{\frac{5}{4}}.$$

(19) For the partition $65 = 3 \cdot 3 + 2 \cdot 7 + 11 + 31$, the corresponding term is

$$\binom{7}{3, 2, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{11\nu} q^{\frac{31}{52}} \cdot (-8) = 26880q^{\frac{5}{4}}.$$

(20) For the partition $65 = 3 \cdot 3 + 2 \cdot 7 + 15 + 27$, the corresponding term is

$$\binom{7}{3, 2, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{10\nu} q^{\frac{27}{52}} = 3360q^{\frac{5}{4}}.$$

(21) For the partition $65 = 3 \cdot 3 + 2 \cdot 7 + 19 + 23$, the corresponding term is

$$\binom{7}{3, 2, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = 80640q^{\frac{5}{4}}.$$

(22) For the partition $65 = 3 \cdot 3 + 3 \cdot 7 + 35$, the corresponding term is

$$\binom{7}{3, 3, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot \zeta^{12\nu} q^{\frac{35}{52}} \cdot 17 = -19040q^{\frac{5}{4}}.$$

(23) For the partition $65 = 3 \cdot 3 + 11 + 3 \cdot 15$, the corresponding term is

$$\binom{7}{3, 1, 3} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^3 = 2240q^{\frac{5}{4}}.$$

(24) For the partition $65 = 3 \cdot 3 + 2 \cdot 11 + 15 + 19$, the corresponding term is

$$\binom{7}{3, 2, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = -26880q^{\frac{5}{4}}.$$

(25) For the partition $65 = 3 \cdot 3 + 3 \cdot 11 + 23$, the corresponding term is

$$\binom{7}{3, 3, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^3 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^3 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = -6720q^{\frac{5}{4}}.$$

(26) For the partition $65 = 2 \cdot 3 + 4 \cdot 7 + 31$, the corresponding term is

$$\binom{7}{2, 4, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^4 \cdot \zeta^{11\nu} q^{\frac{31}{52}} \cdot (-8) = -13440q^{\frac{5}{4}}.$$

(27) For the partition $65 = 2 \cdot 3 + 3 \cdot 7 + 2 \cdot 19$, the corresponding term is

$$\binom{7}{2, 3, 2} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot (\zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8))^2 = 107520q^{\frac{5}{4}}.$$

(28) For the partition $65 = 2 \cdot 3 + 3 \cdot 7 + 15 + 23$, the corresponding term is

$$\binom{7}{2, 3, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = -40320q^{\frac{5}{4}}.$$

(29) For the partition $65 = 2 \cdot 3 + 3 \cdot 7 + 11 + 27$, the corresponding term is

$$\binom{7}{2, 3, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{10\nu} q^{\frac{27}{52}} = 6720q^{\frac{5}{4}}.$$

(30) For the partition $65 = 2 \cdot 3 + 2 \cdot 7 + 3 \cdot 15$, the corresponding term is

$$\binom{7}{2, 2, 3} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^3 = -6720q^{\frac{5}{4}}.$$

(31) For the partition $65 = 2 \cdot 3 + 2 \cdot 7 + 11 + 15 + 19$, the corresponding term is

$$\binom{7}{2, 2, 1, 1, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = 161280q^{\frac{5}{4}}.$$

(32) For the partition $65 = 2 \cdot 3 + 2 \cdot 7 + 2 \cdot 11 + 23$, the corresponding term is

$$\binom{7}{2, 2, 2, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = 60480q^{\frac{5}{4}}.$$

(33) For the partition $65 = 2 \cdot 3 + 7 + 2 \cdot 11 + 2 \cdot 15$, the corresponding term is

$$\binom{7}{2, 1, 2, 2} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^2 = 20160q^{\frac{5}{4}}.$$

(34) For the partition $65 = 2 \cdot 3 + 7 + 3 \cdot 11 + 19$, the corresponding term is

$$\binom{7}{2, 1, 3, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^3 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = -53760q^{\frac{5}{4}}.$$

(35) For the partition $65 = 2 \cdot 3 + 4 \cdot 11 + 15$, the corresponding term is

$$\binom{7}{2, 4, 1} (\zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1))^2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^4 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) = -3360q^{\frac{5}{4}}.$$

(36) For the partition $65 = 3 + 5 \cdot 7 + 27$, the corresponding term is

$$\binom{7}{1, 5, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^5 \cdot \zeta^{10\nu} q^{\frac{27}{52}} = -1344q^{\frac{5}{4}}.$$

(37) For the partition $65 = 3 + 4 \cdot 7 + 15 + 19$, the corresponding term is

$$\binom{7}{1, 4, 1, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^4 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = -53760q^{\frac{5}{4}}.$$

(38) For the partition $65 = 3 + 4 \cdot 7 + 11 + 23$, the corresponding term is

$$\binom{7}{1, 4, 1, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^4 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = -40320q^{\frac{5}{4}}.$$

(39) For the partition $65 = 3 + 3 \cdot 7 + 11 + 2 \cdot 15$, the corresponding term is

$$\binom{7}{1, 3, 1, 2} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^2 = -26880q^{\frac{5}{4}}.$$

(40) For the partition $65 = 3 + 3 \cdot 7 + 2 \cdot 11 + 19$, the corresponding term is

$$\binom{7}{1, 3, 2, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = 107520q^{\frac{5}{4}}.$$

(41) For the partition $65 = 3 + 2 \cdot 7 + 3 \cdot 11 + 15$, the corresponding term is

$$\binom{7}{1, 2, 3, 1} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^3 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) = 26880q^{\frac{5}{4}}.$$

(42) For the partition $65 = 3 + 7 + 5 \cdot 11$, the corresponding term is

$$\binom{7}{1, 1, 5} \zeta^{4\nu} q^{\frac{3}{52}} \cdot (-1) \cdot \zeta^{5\nu} q^{\frac{7}{52}} \cdot 2 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^5 = -2688q^{\frac{5}{4}}.$$

(43) For the partition $65 = 6 \cdot 7 + 23$, the corresponding term is

$$\binom{7}{6, 1} (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^6 \cdot \zeta^{9\nu} q^{\frac{23}{52}} \cdot 6 = 2688q^{\frac{5}{4}}.$$

(44) For the partition $65 = 5 \cdot 7 + 2 \cdot 15$, the corresponding term is

$$\binom{7}{5, 2} (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^5 \cdot (\zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2))^2 = 2688q^{\frac{5}{4}}.$$

(45) For the partition $65 = 5 \cdot 7 + 11 + 19$, the corresponding term is

$$\binom{7}{5, 1, 1} (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^5 \cdot \zeta^{6\nu} q^{\frac{11}{52}} \cdot 2 \cdot \zeta^{8\nu} q^{\frac{19}{52}} \cdot (-8) = -21504q^{\frac{5}{4}}.$$

(46) For the partition $65 = 4 \cdot 7 + 2 \cdot 11 + 15$, the corresponding term is

$$\binom{7}{4, 2, 1} (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^4 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^2 \cdot \zeta^{7\nu} q^{\frac{15}{52}} \cdot (-2) = -13440q^{\frac{5}{4}}.$$

(47) For the partition $65 = 3 \cdot 7 + 4 \cdot 11$, the corresponding term is

$$\binom{7}{4, 3} (\zeta^{5\nu} q^{\frac{7}{52}} \cdot 2)^3 \cdot (\zeta^{6\nu} q^{\frac{11}{52}} \cdot 2)^4 = 4480q^{\frac{5}{4}}.$$

Hence, for $\Phi_{42}(x_1(z), \dots, x_6(z))$ which is a modular form for $\Gamma(1)$ with weight 42, the lowest degree term is given by

$$\begin{aligned} & (-154 - 1092 + 1428 - 2688 - 252 + 14280 + 6720 - 3360 + 7560 - 840 + \\ & - 20160 - 3360 - 13440 + 2520 - 107520 + 40320 - 3360 + 26880 + 26880 + 3360 + \\ & + 80640 - 19040 + 2240 - 26880 - 6720 - 13440 + 107520 - 40320 + 6720 - 6720 + \\ & + 161280 + 60480 + 20160 - 53760 - 3360 - 1344 - 53760 - 40320 - 26880 + \\ & + 107520 + 26880 - 2688 + 2688 + 2688 - 21504 - 13440 + 4480)q^{\frac{5}{4}} \cdot q^{\frac{42}{24}} \\ & = 226842q^3. \end{aligned}$$

Thus,

$$\Phi_{42}(x_1(z), \dots, x_6(z)) = q^3(13 \cdot 226842 + O(q)).$$

The leading term of $\Phi_{42}(x_1(z), \dots, x_6(z))$ together with its weight 42 suffice to identify this modular form with

$$\Phi_{42}(x_1(z), \dots, x_6(z)) = 13 \cdot 226842 \Delta(z)^3 E_6(z).$$

Up to a constant, we revise the definition of Φ_{32} , Φ_{42} and Φ_{44} :

$$\Phi_{32} = -\frac{1}{13 \cdot 1840} \left(\sum_{\nu=0}^{12} w_{\nu}^8 + w_{\infty}^8 \right), \quad \Phi_{42} = \frac{1}{13 \cdot 226842} \left(\sum_{\nu=0}^{12} \delta_{\nu}^7 + \delta_{\infty}^7 \right), \quad (4.1)$$

$$\Phi_{44} = \frac{1}{13 \cdot 146905} \left(\sum_{\nu=0}^{12} w_{\nu}^{11} + w_{\infty}^{11} \right). \quad (4.2)$$

Consequently,

$$\begin{cases} \Phi_{32}(x_1(z), \dots, x_6(z)) = \eta(z)^8 \Delta(z)^2 E_4(z), \\ \Phi_{42}(x_1(z), \dots, x_6(z)) = \Delta(z)^3 E_6(z), \\ \Phi_{44}(x_1(z), \dots, x_6(z)) = \eta(z)^8 \Delta(z)^3 E_4(z). \end{cases} \quad (4.3)$$

From now on, we will use the following abbreviation $\Phi_j = \Phi_j(x_1(z), \dots, x_6(z))$ for $j = 32, 42$ and 44 . The relations

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = \frac{\Phi_{32}^3}{\Phi_{12}^8} = \frac{\Phi_{44}^3}{\Phi_{12}^{11}}, \quad j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = \frac{\Phi_{42}^2}{\Phi_{12}^7} \quad (4.4)$$

together with (3.20) give the equations

$$\begin{cases} \Phi_{20}^3 \Phi_{12}^2 - \Phi_{42}^2 = 1728 \Phi_{12}^7, \\ \Phi_{32}^3 - \Phi_{12}^5 \Phi_{18}^2 = 1728 \Phi_{12}^8, \\ \Phi_{32}^3 - \Phi_{12}^3 \Phi_{30}^2 = 1728 \Phi_{12}^8, \\ \Phi_{32}^3 - \Phi_{12} \Phi_{42}^2 = 1728 \Phi_{12}^8, \\ \Phi_{44}^3 - \Phi_{12}^8 \Phi_{18}^2 = 1728 \Phi_{12}^{11}, \\ \Phi_{44}^3 - \Phi_{12}^6 \Phi_{30}^2 = 1728 \Phi_{12}^{11}, \\ \Phi_{44}^3 - \Phi_{12}^4 \Phi_{42}^2 = 1728 \Phi_{12}^{11}. \end{cases} \quad (4.5)$$

This leads to the modular parametrizations of the following three exceptional singularities:

$$Q_{18} : x^3 + y^8 + yz^2 = 0, \quad E_{20} : x^3 + y^{11} + z^2 = 0, \quad x^7 + x^2 y^3 + z^2 = 0.$$

Note that Q_{18} and E_{20} are two singularities in the pyramids of 14 exceptional singularities of modalities 2 (see [Ar4], p.255). However, the last singularity does not appear in the singularities with the number of moduli $m = 0, 1$ and 2 (see [Ar4]). Moreover, the last three equations in (4.5) give a new analytic construction of solutions for the Diophantine equation $x^p + y^q = z^r$ in the case $(p, q, r) = (2, 3, 11)$. Thus, we complete the proof of Theorem 1.2 and Corollary 1.3. □

It should be pointed out that in [K], [K1], [K2], [K3], [KF1], Klein connected the simple groups $\mathrm{PSL}(2, 5)$ and $\mathrm{PSL}(2, 7)$ with singularities E_8 and E_{12} , respectively. However, he could not connect the simple group $\mathrm{PSL}(2, 11)$ with any singularities (see [K4], [KF2], p.413), especially the singularity E_{20} . By Theorem 1.2, we connect the simple group $\mathrm{PSL}(2, 13)$ with singularity E_{20} and solve this problem dating back to the works of Klein. Moreover, by Theorem 1.1 and Theorem 1.2, we connect the simple group $\mathrm{PSL}(2, 13)$ with many singularities: E_8, Q_{18}, E_{20} and $x^7 + x^2y^3 + z^2 = 0$.

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